Intuitionistic Propositional Logic

CIS700 — Fall 2023

Kris Micinski

Natural Deduction is a logical reasoning system which defines an explicit syntactic notion of proofs at the same time as the propositions they inhabit.

This mindset is closely related to the notion of type-checking a program, wherein subexpressions types are surmised and compositional rules allow you to combine subexpressions in a well-typed manner.

In natural deduction, we say “the follow rules are the **only** ways in which a proof of a logical statement is built.”

To say something is true **is the same as** having a ***syntactic, materialized*** proof for it

My lecture very closely follows Frank Pfenning’s notes from his various classes, I will link these notes in the Google Group.

I am not a researcher in type theory (not my branch of PL!), so I will defer to expert expositions when necessary.

“Intuitionistic logic, sometimes more generally called constructive logic, refers to systems of symbolic logic that differ from the systems used for classical logic by more closely mirroring the notion of constructive proof. — Wikipedia”

Intuitionism is the notion of identifying a **true statement** with a **symbolic proof of that statement**

Last lecture, we talked about a model (i.e., set)-theoretic perspective, mapping variables to values. This has issues in handling higher-order objects (Russel’s paradox) which do not crop up in the propositional setting—but the study of higher-order logic (wherein one can quantify over propositions) motivated the study of **intuitionistic type theory**

Introduction and Elimination Forms

Proofs of statements in intuitionistic logic discuss the formation of connectives. In classical logic, we typically construe connectives as encodings into a minimal form (e.g., CNF/DNF). This pushes reasoning into a set-theoretic interpretation.

Specifically problematic for computers: explicitly representing an interpretation (e.g., as a set) may be either (a) intractable or (b) impossible due to infiniteness.

By contrast, intuitionism dictates that when we discuss the meaning of a connective, we completely define a set of **formation rules** which break down into two broad categories:

- **Introduction Forms** — a connective **appears new** in a conclusion
- **Elimination Forms** — a connective is **consumed** and disappears in the conclusion

The **introduction** form for and ($\wedge$) is a proof schema which tells us how we can introduce $\wedge$ s into a conclusion.

$$\wedge\mathbf{I}\ \frac{P \text{ True} \quad Q \text{ True}}{P \wedge Q \text{ True}}$$

There are two **elimination** forms for $\wedge$: the **first** eliminator selects the left item (discarding the second), and the **second** eliminator selects the right (discarding the left)

$$\wedge\mathbf{E1}\ \frac{P \wedge Q \text{ True}}{P \text{ True}} \qquad \wedge\mathbf{E2}\ \frac{P \wedge Q \text{ True}}{Q \text{ True}}$$

A crucial problem — the need for premises

Let's say we want to write proofs of true statements involving ∧. This is the kind of thing we should be able to do now that we've defined the introduction and elimination forms for ∧.

Unfortunately, this doesn't work. Look at this:

$$\dfrac{\dfrac{A \wedge (B \wedge C) \text{ True}}{(B \wedge C) \text{ True}}\ \wedge\mathbf{E2}}{B \text{ True}}\ \wedge\mathbf{E1}$$

The **reasoning** here works, but following this reasoning allows us to conclude that an arbitrary proposition is true. Obviously, there are some false statements (A ∧ ¬A), so there **must** be a problem!

A crucial problem — the need for premises

This is **not** a proof, it is a suppositional line of reasoning! We have **assumed** that $A \wedge (B \wedge C)$ True, and used that to derive B True

Intuitionistic logic gives **names** to assumptions. We will reject this as a "proof" because the hypothesis is not explicitly introduced. We will do this by introducing them into an **environment**, which allows naming hypotheses

Hypotheses get introduced (and **named**) by the introduction of ⇒.

To prove A ⇒ B, we assume A (by introducing it as a named hypothesis, which may then be referenced) and showing B:

Named hypothesis

$$\dfrac{}{A \text{ True}}\,u$$

$$\cdots$$

$$\dfrac{B \text{ True}}{A \Rightarrow B \text{ True}} \Rightarrow\!\mathbf{I}$$

(End of IPL, day 1)

We will cover more intuitionistic logic next week