Conflict-Directed Clause Resolution (CDCL); Part 1

CIS700 — Fall 2023

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This week we'll study CDCL, the state-of-the-art algorithm for SAT solving. CDCL extends DPLL to ***learn*** derived clauses that capture the "root cause" of conflicts, allowing the solver to generalize conflicts to "lemmas" which help cull the state space.

As we will see, these learned clauses (lemmas) are added in a "sound" way that do not change satisfiability, but help *accelerate* the search.

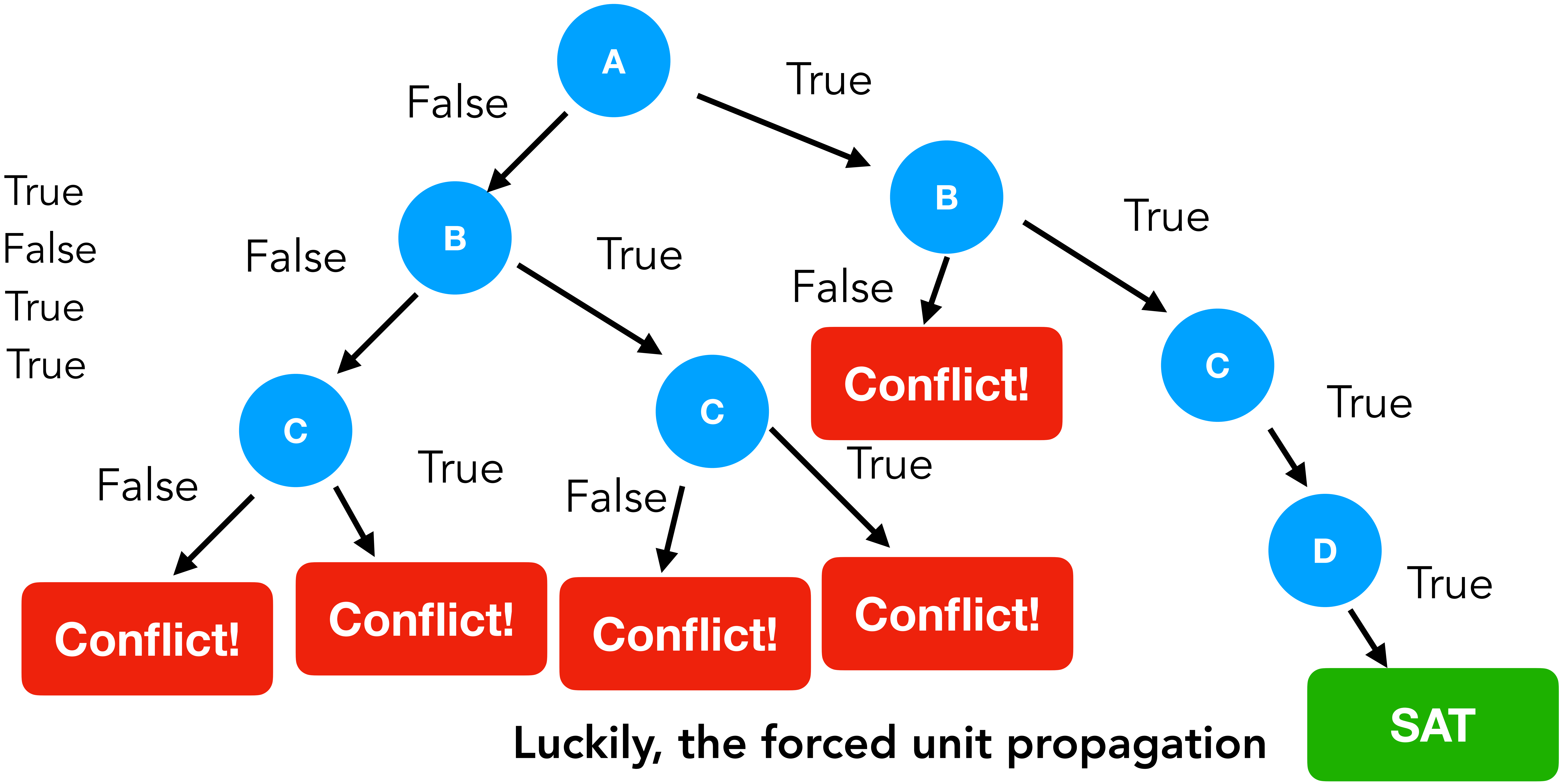
The CDCL algorithm has a lot of moving parts, and really takes some time to understand—we will spend a few days discussing.

Last Lecture: DPLL

Main idea: *search-based* algorithm for materializing an enumeration. Very space efficient, but time? Leaves a lot to be desired. Early-day computers are space pressured.

¬A ∨ B ∨ C
A ∨ C ∨ D
A ∨ C ∨ ¬D
A ∨ ¬C ∨ D
A ∨ ¬C ∨ ¬D
¬B ∨ ¬C ∨ D
¬A ∨ B ∨ ¬C
¬A ∨ ¬B ∨ C

A = True
B = False
C = True
D = True



Luckily, the forced unit propagation D=True leads us to success!

The basic structure of CDCL is similar to DPLL:

- Perform ***unit propagation*** (i.e., "forced assignments") until it is not possible to do so anymore
- If you hit a contradiction in doing *just unit propagation* then you return UNSAT—no guessing was involved
- Now, in a loop, guess an assignment:
  - *Decide* an undecided variable
  - Unit propagate to a fixed point (until you can't anymore)
  - When you reach a conflict:
    - ***Learn*** clause that captures the "reason" for the conflict
    - ***Backtrack*** *non-chronologically* in a way informed by the conflict, to avoid reaching the same conflict again

Reminder: Unit Propagation

Unit propagation (boolean constraint propagation) is the idea that (unsatisfied) clauses containing only **one** unassigned literal are "forced" to have that value in any satisfying assignment.

For example, consider the clause $x_3 \vee \neg x_5 \vee \neg x_7 \vee x_8$, and the current partial assignment x_3=F x_5=T, x_8=F, now $\neg x_7$ is a *logical consequence* of the current valuation whenever the clause holds.

Similarly, if there is another clause: $\neg x_5 \vee x_7 \vee x_9$, unit propagation of $\neg x_7$ *now* forces the assignment of x_9 (as everything else if false)

Decision Levels

DPLL and CDCL both **alternate between BCP and decision (guessing)**

Building on the intuition from the last slide, it is sensible to construe the “decision level” as the number of *decision literals* in a trace.

So when we refer to the “decision level” of the CDCL algorithm, we are saying: how many ***genuine guesses*** have we made up until this point—ignoring the (possibly large) number of forced implications due to iterative application of BCP

Exercise: Boolean Constraint Propagation

In sum, boolean constraint propagation performs **one-step transitive** reasoning to derived forced implications.

For the following SAT instance, show how iterated application of unit propagation will lead to a conflict:

$$(1)\ \neg x_3 \lor x_5 \lor x_6$$
$$(2)\ x_3 \lor x_5$$
$$(3)\ \neg x_5$$
$$(4)\ \neg x_6$$

Exercise Solution…

$$(1)\ \neg x_3 \lor x_5 \lor x_6$$
$$(2)\ x_3 \lor x_5$$
$$(3)\ \neg x_5$$
$$(4)\ \neg x_6$$

One possible solution: $\neg x_5$ forces x_5=False. Now perform BCP: this obtains x_3 from (2) and thus x_3=True. Now, from (4) conclude x_6=False. BCP of $\neg x_6$ on (1) yields $\neg x_3$, which gives a conflict with the previously-derived x_3=True (from BCP of 2 on $\neg x_5$).

The trail

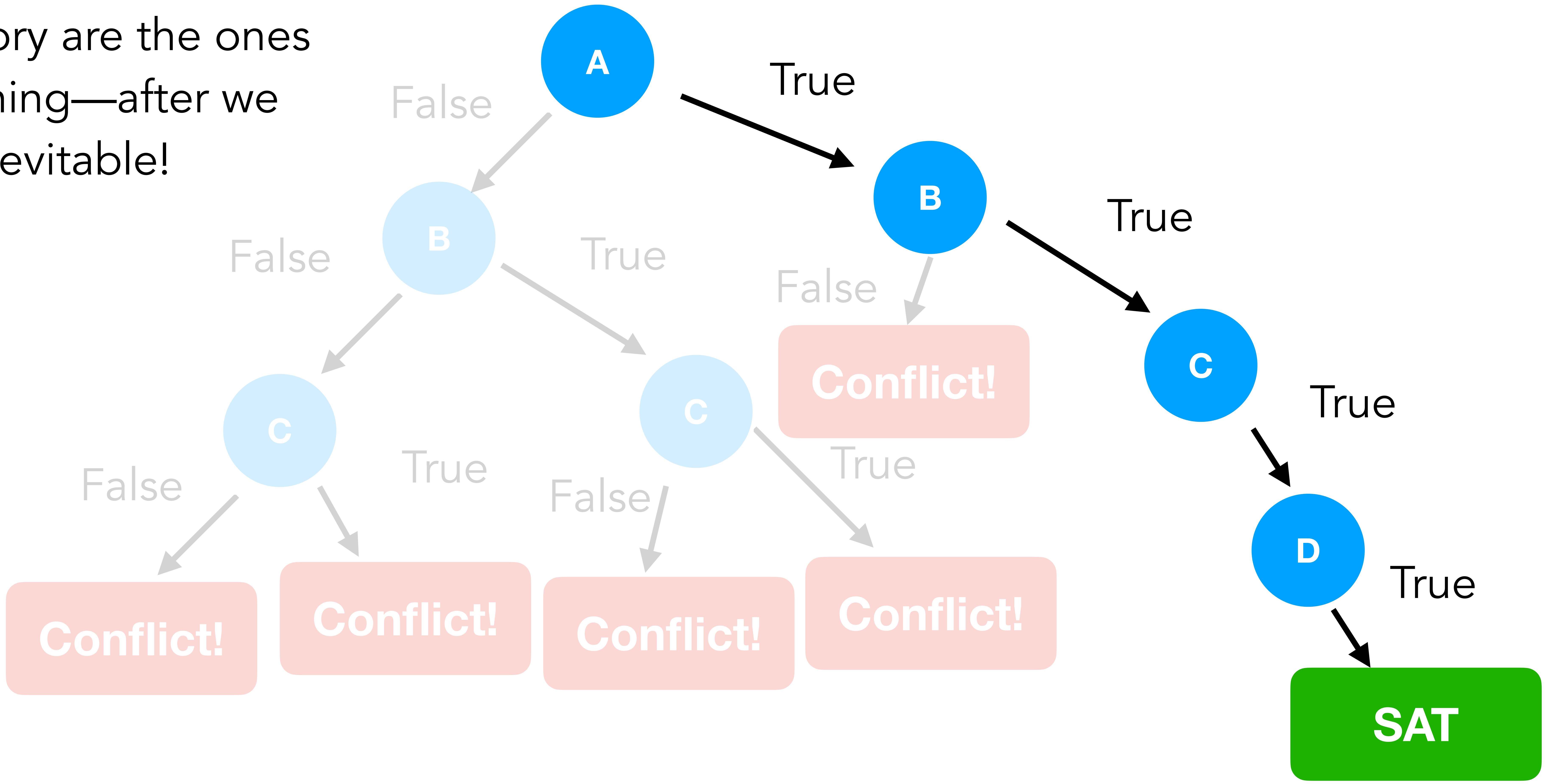
A key data structure in the CDCL algorithm is the notion of a **trail**, a LIFO list of decisions along side their provenance. The trail accumulates a partial assignment and captures (a part of) the solver's execution at each step in the computation.

The trail is a list of literals, each of these literals is annotated with either (a) a special **dec** token, indicating that the reason for the literal's inclusion was a decision, (b) a unit clause present in the instance, or (c) a pointer to the clause that forced propagation via BCP

The trail is useful because it separates the solver's decision points from its unit propagations. Later on, we will backtrack to decision points, throwing away the unit propagations after a (wrong) decision.

The trail records the "focus" into the tree of DPLL, but separates (a) possibly-bad guesses (decision nodes) and (b) forced consequences of those decisions.

One of CDCL's key insights is that the decisions in the (a) category are the ones we really need to be tuning—after we choose (a), (b) is inevitable!



The trail

When we write a trail, we may also include the decision level via "@N" where N is the decision level. Thus, our trails will have the form

$l\_i@0, \ldots, l_k^{dec}, l_j @ 1, \ldots., l_{k+1}^{dec}, l_{j+1} @ 2, \ldots$

I.e., an initial set of unit clauses and their propagations, followed by decisions with (possibly zero) consequences of unit propagation

Conflicts from BCP at level 0 represent UNSAT problems that require **no guessing to prove UNSAT**

These represent "easy" UNSAT instances, no guessing involved!

Example trail

$$(1)\ \neg x_3 \lor x_5 \lor x_6$$

$$(2)\ x_3 \lor x_5$$

$$(3)\ \neg x_5$$

$$(4)\ \neg x_6$$

For the problem here, the trail looks like the following:

$\neg x_5 @0^{(3)}, x_3 @0^{(2_5)}, \neg x_6 @0^{(4)}, \neg x_3 @0^{(1)}$, **CONFLICT**

Now an example which *is* satisfiable

Consider the following clauses:

(1) $x_0 \vee x_3 \vee \neg x_4$

(2) $\neg x_3 \vee x_5$

(3) $x_4 \vee x_3$

No unit clauses, and thus we must guess. We are guessing at **decision-level zero**, and we will guess x_3 = True, our trail looks like

$x_3@0^{dec}$

Now an example which *is* satisfiable

Consider the following clauses:

(1) $x_0 \vee x_3 \vee \neg x_4$

(2) $\neg x_3 \vee x_5$

(3) $x_4 \vee x_3$

Now, we need to perform BCP for x_3, we always eagerly apply BCP. Now (2) tells us we need to decide x_5, and our trail looks like:

$x_3@0^{dec}, x_5@1^{(2)}$

Now an example which *is* satisfiable

Consider the following clauses:

(1) $x_0 \vee x_3 \vee \neg x_4$

(2) $\neg x_3 \vee x_5$

(3) $x_4 \vee x_3$

Now we can't apply BCP anymore (all clauses satisfied) , we have unassigned values, but their values are arbitrary (1/2/3 already satisfied):

$x_3@0^{dec}$, $x_5@1^{(2)}$, $x_0@1^{dec}$, $\neg x_4@2^{dec}$

This gives us a satisfying assignment: x_0=T, x_3=T, x_5=T, x_4=F

Algorithm 1 Typical CDCL algorithm

```
CDCL(φ, ν)
if (UnitPropagation(φ, ν) == CONFLICT)
   then return UNSAT
dl ← 0                                          ▷ Decision level
while (not AllVariablesAssigned(φ, ν))
      do (x, v) = PickBranchingVariable(φ, ν)               ▷ Decide stage
         dl ← dl + 1                            ▷ Increment decision level due to new decision
         ν ← ν ∪ {(x, v)}
         if (UnitPropagation(φ, ν) == CONFLICT)             ▷ Deduce stage
           then β = ConflictAnalysis(φ, ν)                 ▷ Diagnose stage
                if (β < 0)
                  then return UNSAT
                  else Backtrack(φ, ν, β)
                       dl ← β                   ▷ Decrement decision level due to backtracking
return SAT
```

[Joao Marques-Silva, Ines Lynce and Sharad Malik From the "Handbook of Satisfiability]

Implication Graph

Solvers work in terms of linear trails, but it is semantically useful to construe an ***implication graph***, whose vertices are literals and whose edges define the "forced implications" from the rules

Step 1 — Decide

$x_1 \lor x_4$

$x_1 \lor \neg x_3 \lor \neg x_8$

$x_1 \lor \neg x_8 \lor x_{12}$

$x_2 \lor x_{11}$

$\neg x_7 \lor \neg x_3 \lor x_9$

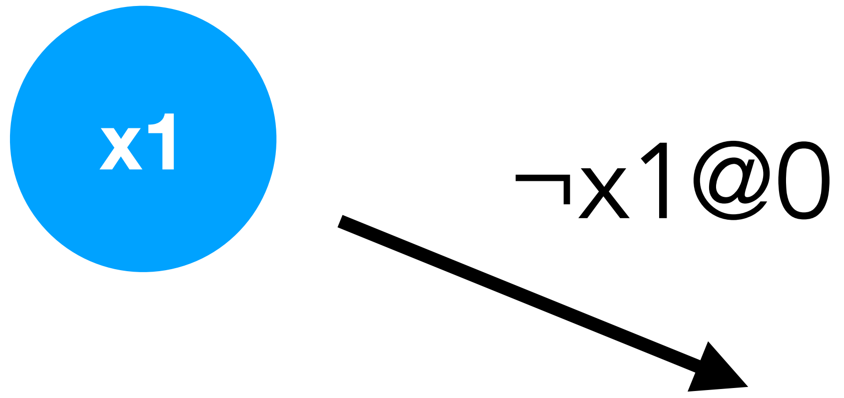
$\neg x_7 \lor x_8 \lor \neg x_9$

$x_7 \lor x_8 \lor \neg x_{10}$

$x_7 \lor x_{10} \lor \neg x_{12}$

No possible unit propagation, thus decide

Arbitrarily, we decide ¬x1



Implication Graph

New node in implication graph, all ***root*** nodes are decisions. Non-root nodes are results of BCP (Bottom right of node labels decision level)

Step 2 — BCP

$x_1 \vee x_4$

$x_1 \vee \neg x_3 \vee \neg x_8$

$x_1 \vee \neg x_8 \vee x_{12}$

$x_2 \vee x_{11}$

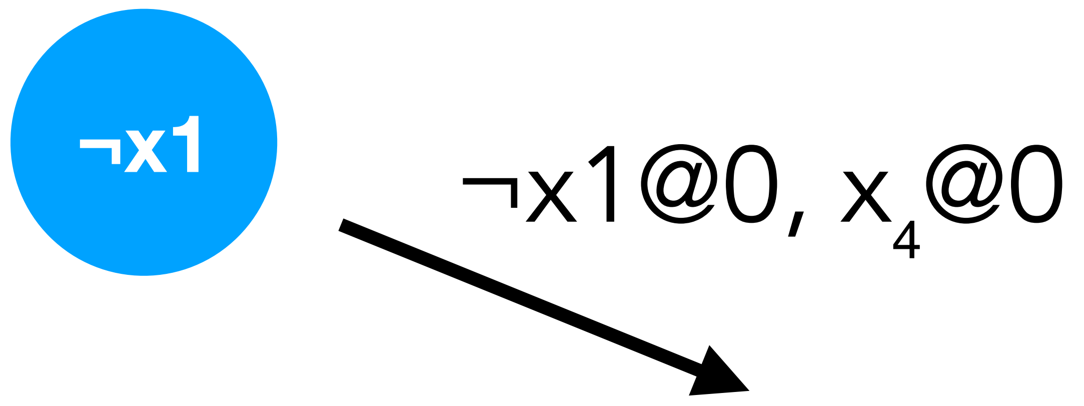
$\neg x_7 \vee \neg x_3 \vee x_9$

$\neg x_7 \vee x_8 \vee \neg x_9$

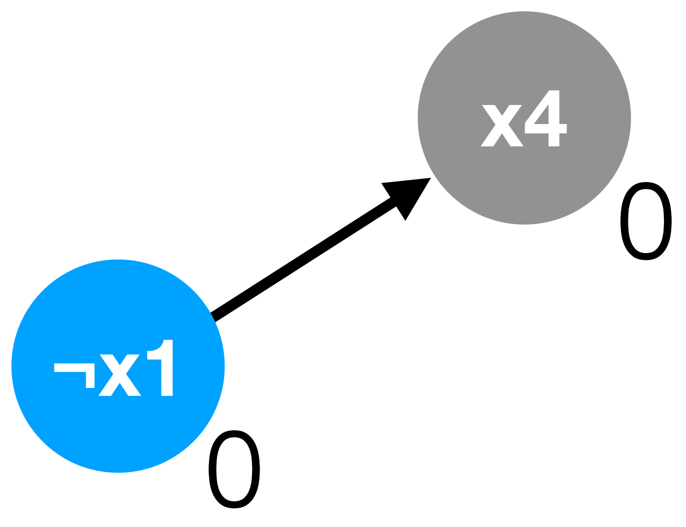
$x_7 \vee x_8 \vee \neg x_{10}$

$x_7 \vee x_{10} \vee \neg x_{12}$

First clause forces x_4, extend trail rooted at x1



Implication Graph



New non-root node x4

Step 3 — Decide Again

$x_1 \lor x_4$

$x_1 \lor \neg x_3 \lor \neg x_8$

$x_1 \lor \neg x_8 \lor x_{12}$

$x_2 \lor x_{11}$

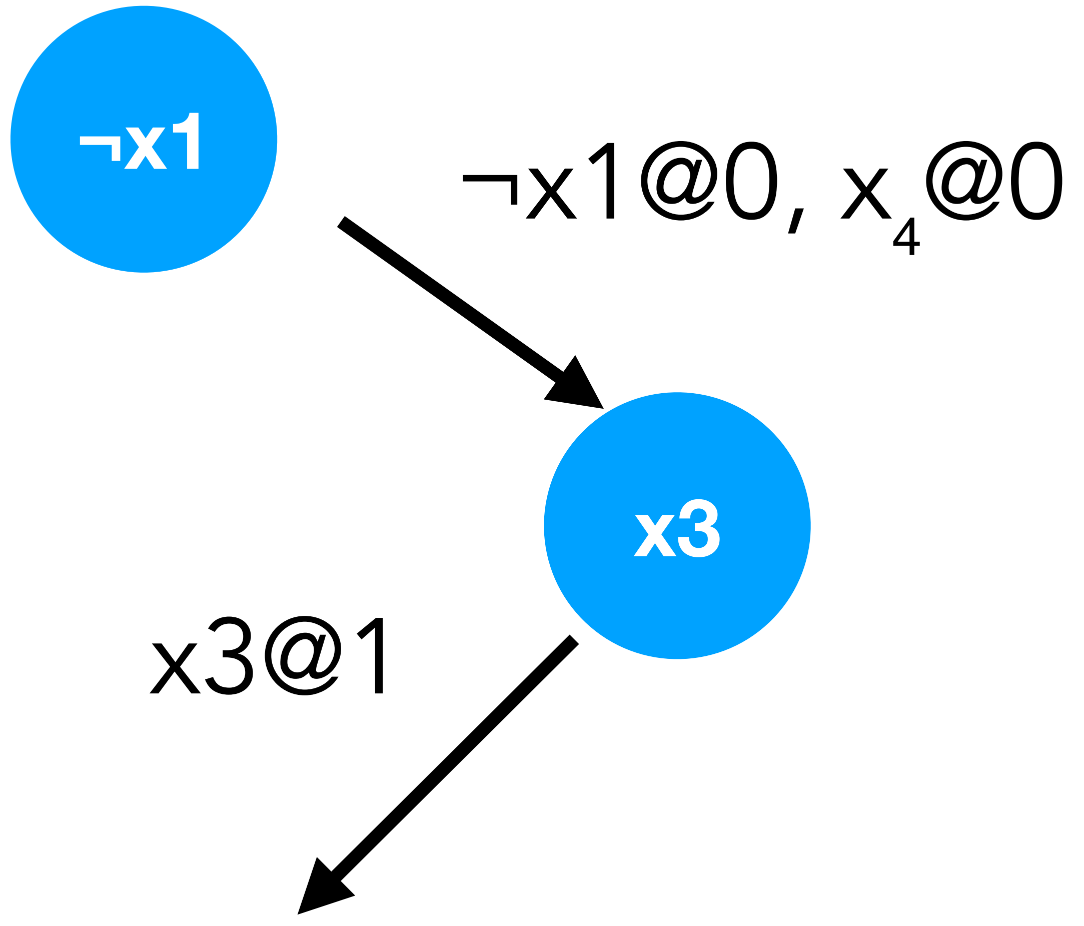
$\neg x_7 \lor \neg x_3 \lor x_9$

$\neg x_7 \lor x_8 \lor \neg x_9$

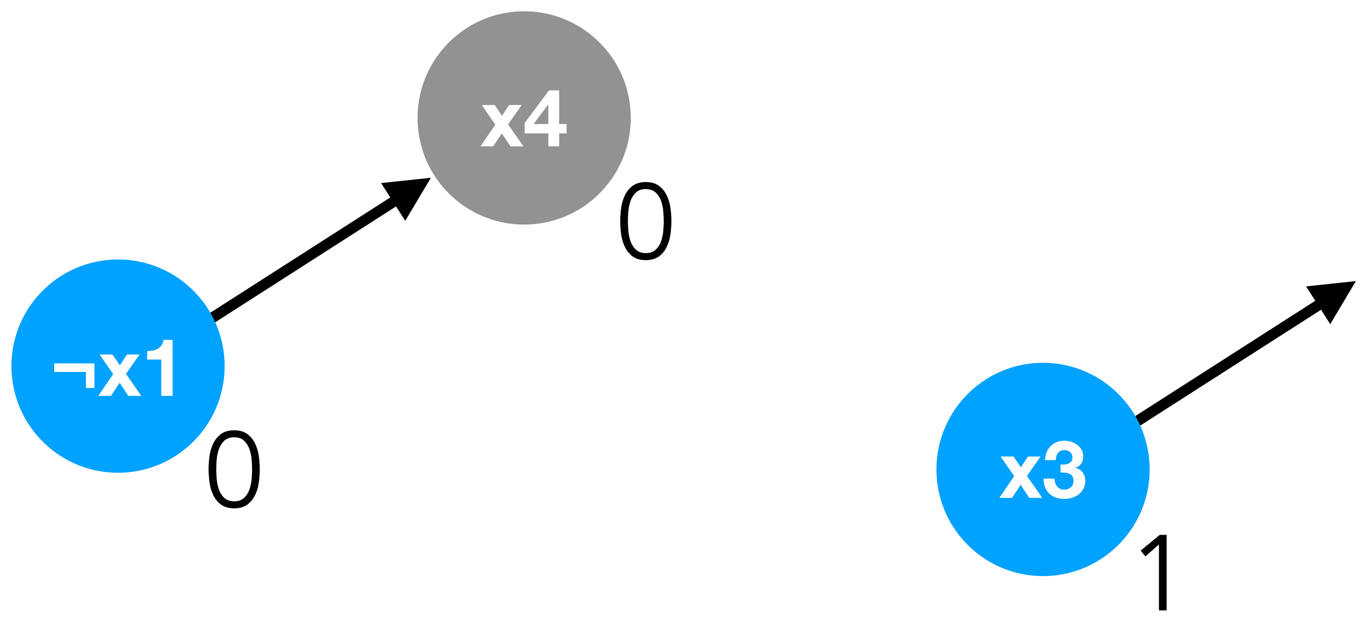
$x_7 \lor x_8 \lor \neg x_{10}$

$x_7 \lor x_{10} \lor \neg x_{12}$

Still more unassigned literals—keep going, next let's assign x3 (True). This is a new decision level



Implication Graph



Step 4 — More BCP

$x_1 \lor x_4$

$x_1 \lor \neg x_3 \lor \neg x_8$

$x_1 \lor \neg x_8 \lor x_{12}$

$x_2 \lor x_{11}$

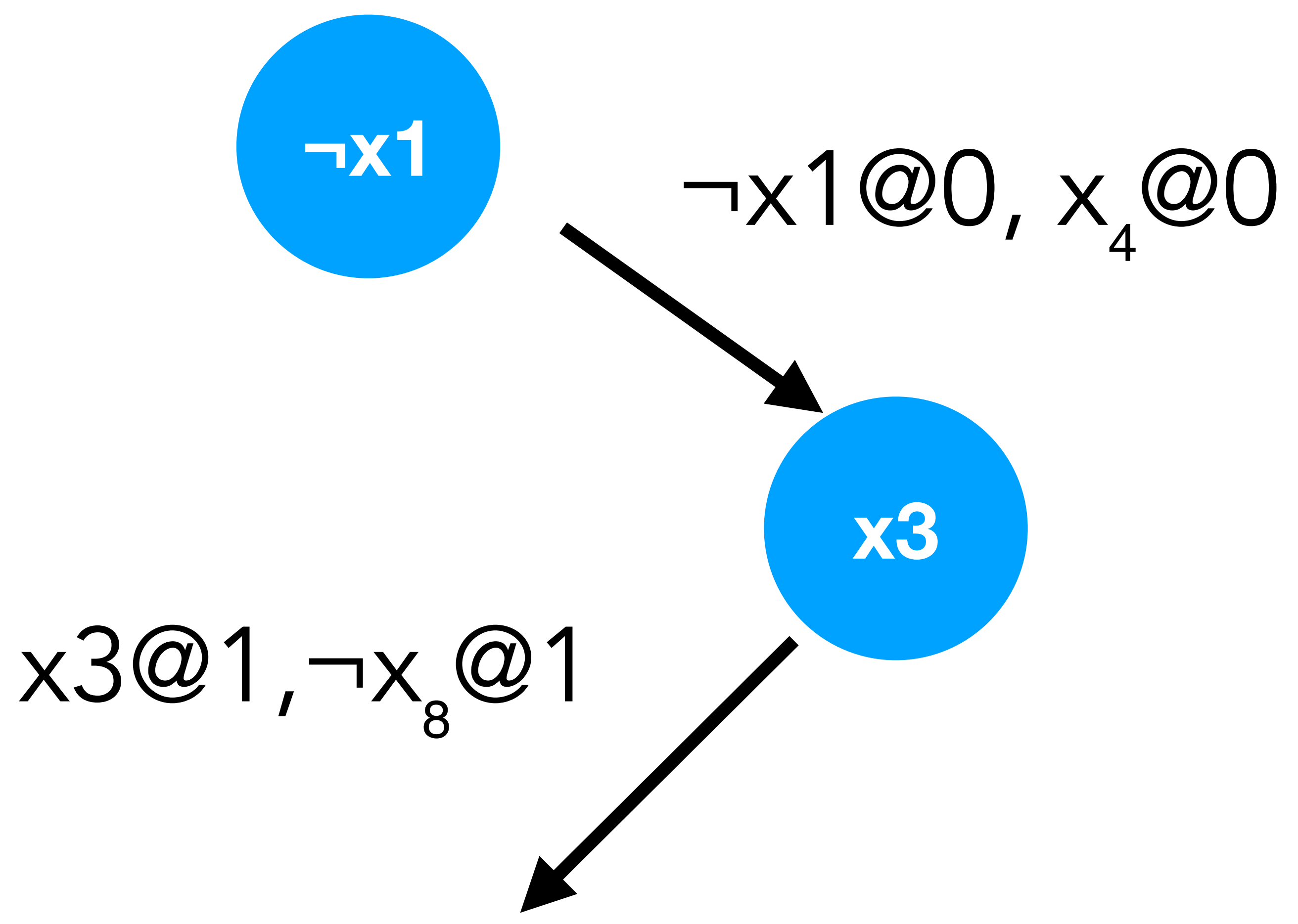
$\neg x_7 \lor \neg x_3 \lor x_9$

$\neg x_7 \lor x_8 \lor \neg x_9$

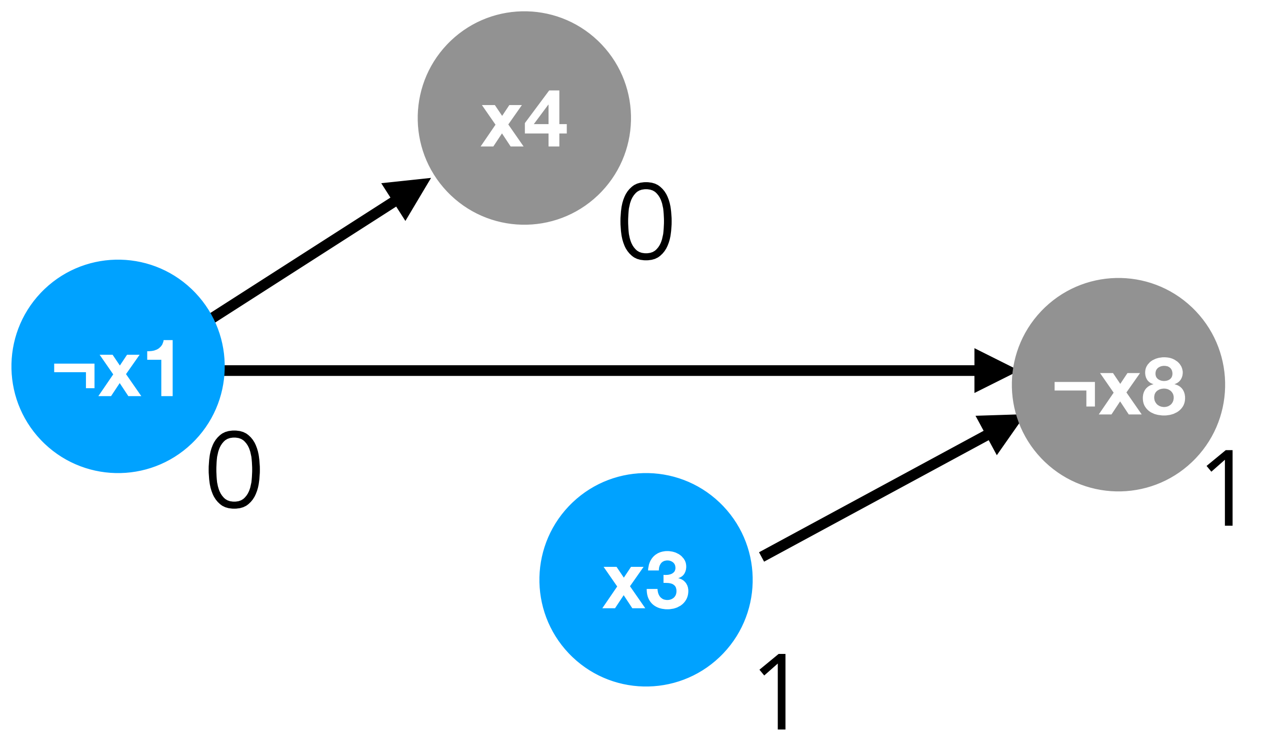
$x_7 \lor x_8 \lor \neg x_{10}$

$x_7 \lor x_{10} \lor \neg x_{12}$

Now x3 is true and x1 is false, thus BCP $\neg x_8$



Implication Graph



Step 5 — Even More BCP

$x_1 \vee x_4$

$x_1 \vee \neg x_3 \vee \neg x_8$

$x_1 \vee \neg x_8 \vee x_{12}$

$x_2 \vee x_{11}$

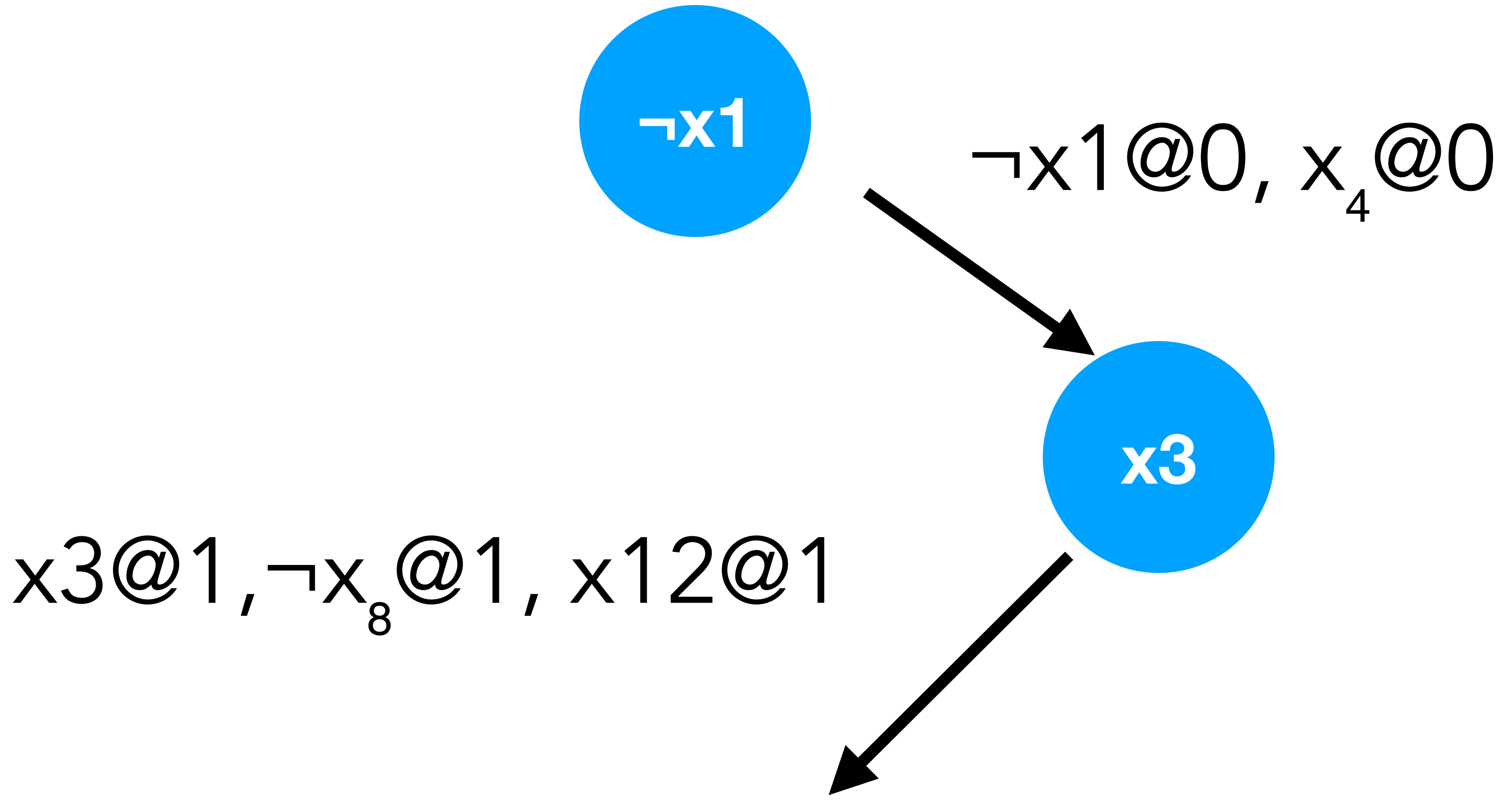
$\neg x_7 \vee \neg x_3 \vee x_9$

$\neg x_7 \vee x_8 \vee \neg x_9$

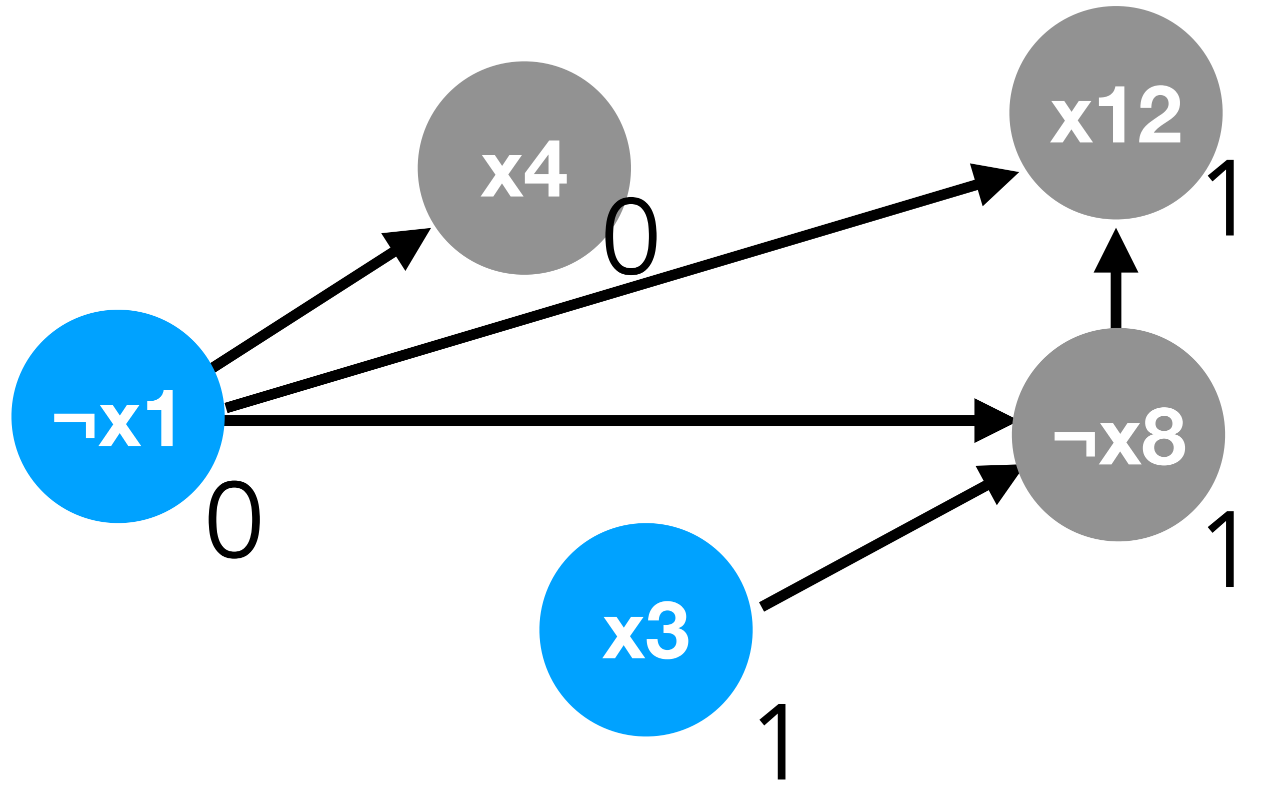
$x_7 \vee x_8 \vee \neg x_{10}$

$x_7 \vee x_{10} \vee \neg x_{12}$

We can now infer x12 from $\neg x_8$



Implication Graph



Step 6 — Back to Guessing

$x_1 \lor x_4$

$x_1 \lor \neg x_3 \lor \neg x_8$

$x_1 \lor \neg x_8 \lor x_{12}$

$x_2 \lor x_{11}$

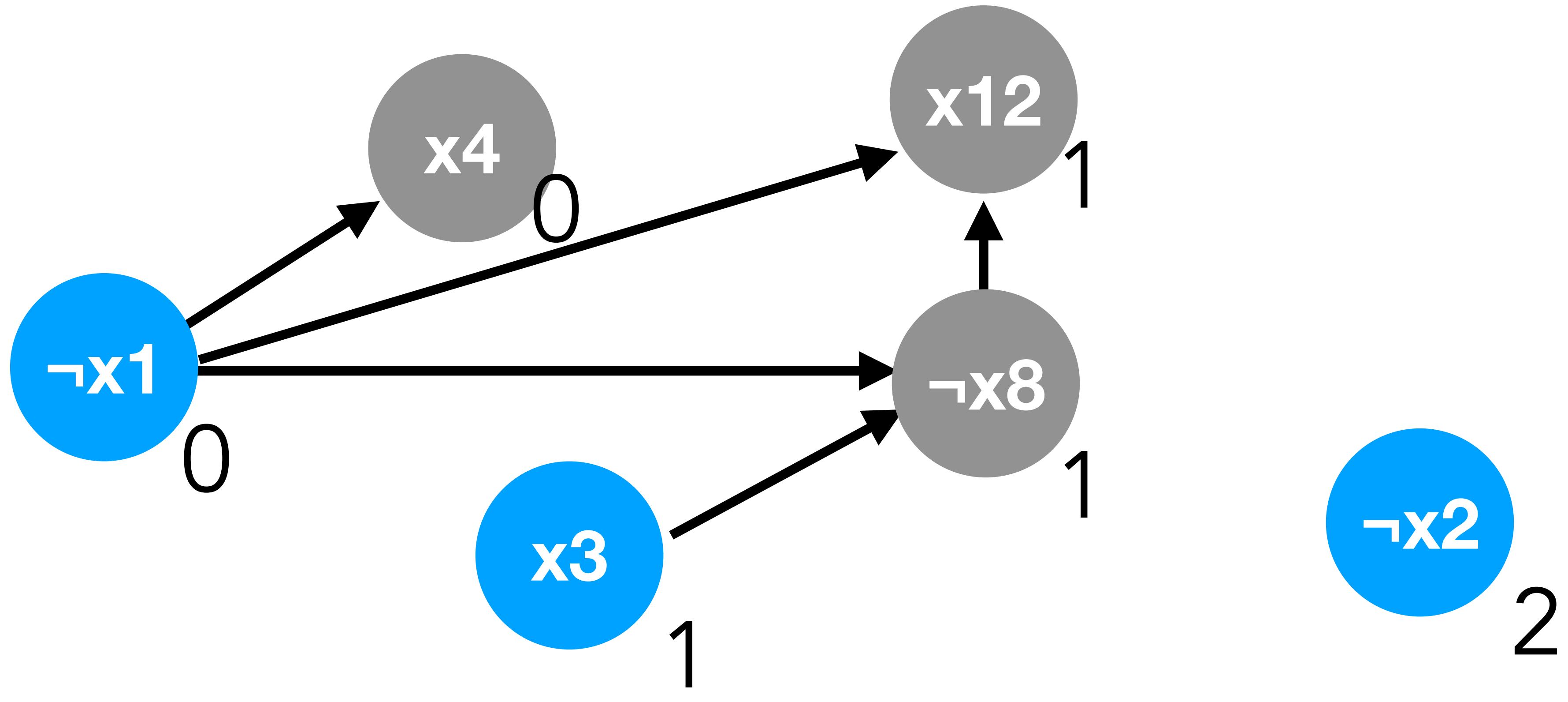
$\neg x_7 \lor \neg x_3 \lor x_9$

$\neg x_7 \lor x_8 \lor \neg x_9$

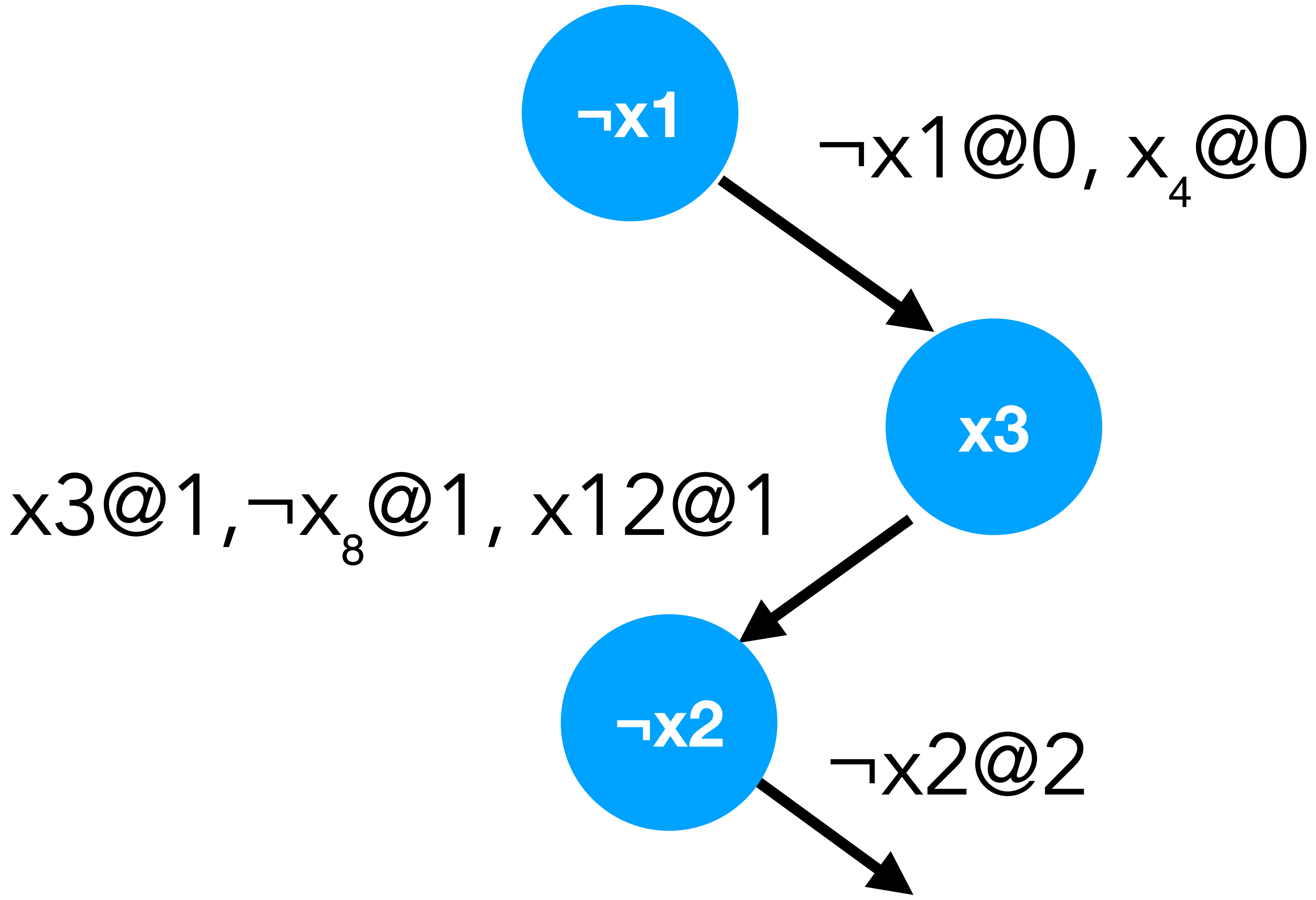
$x_7 \lor x_8 \lor \neg x_{10}$

$x_7 \lor x_{10} \lor \neg x_{12}$

Implication Graph



Now let's guess ¬x2



Step 7 — BCP

$x_1 \vee x_4$

$x_1 \vee \neg x_3 \vee \neg x_8$

$x_1 \vee \neg x_8 \vee x_{12}$

$x_2 \vee x_{11}$

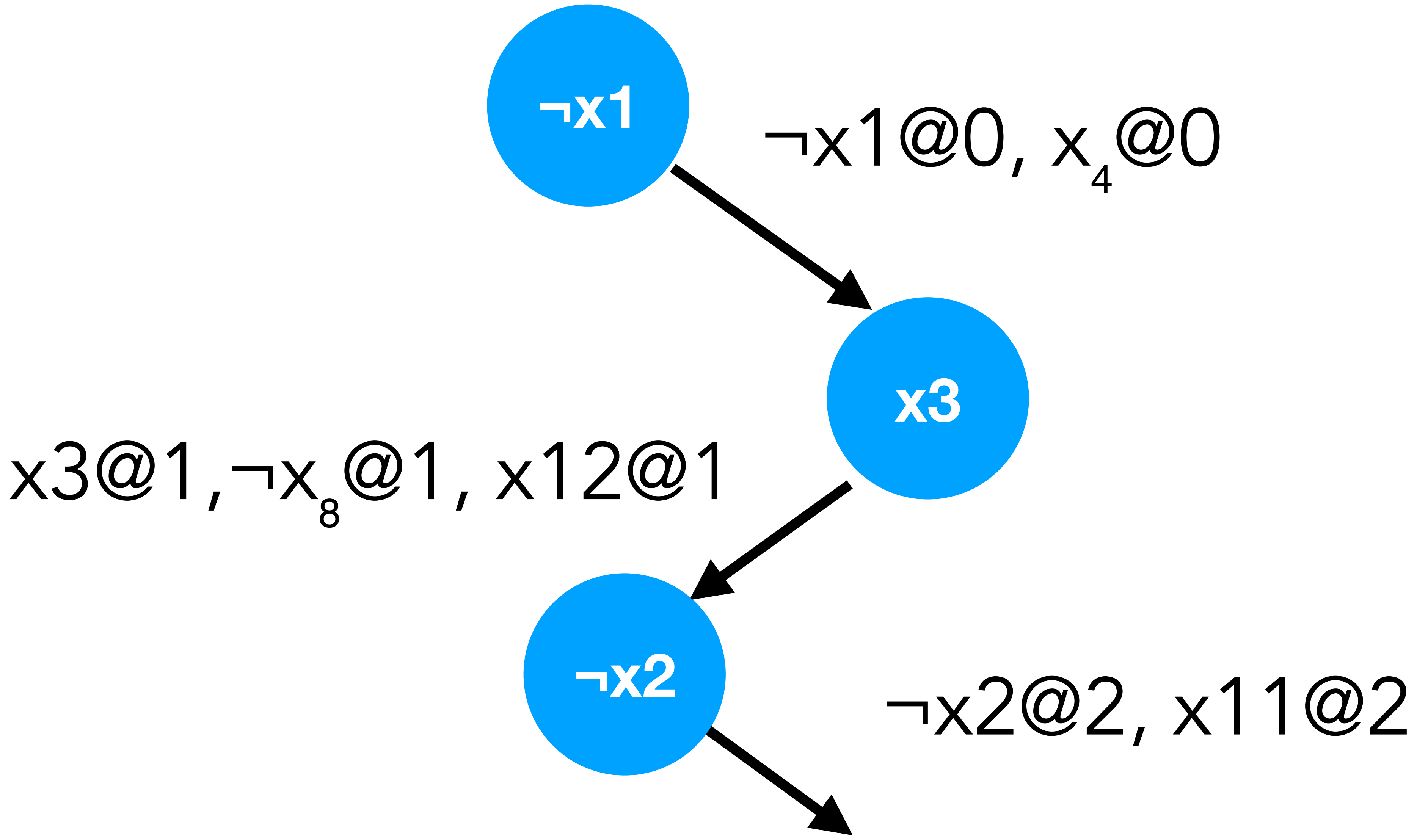
$\neg x_7 \vee \neg x_3 \vee x_9$

$\neg x_7 \vee x_8 \vee \neg x_9$

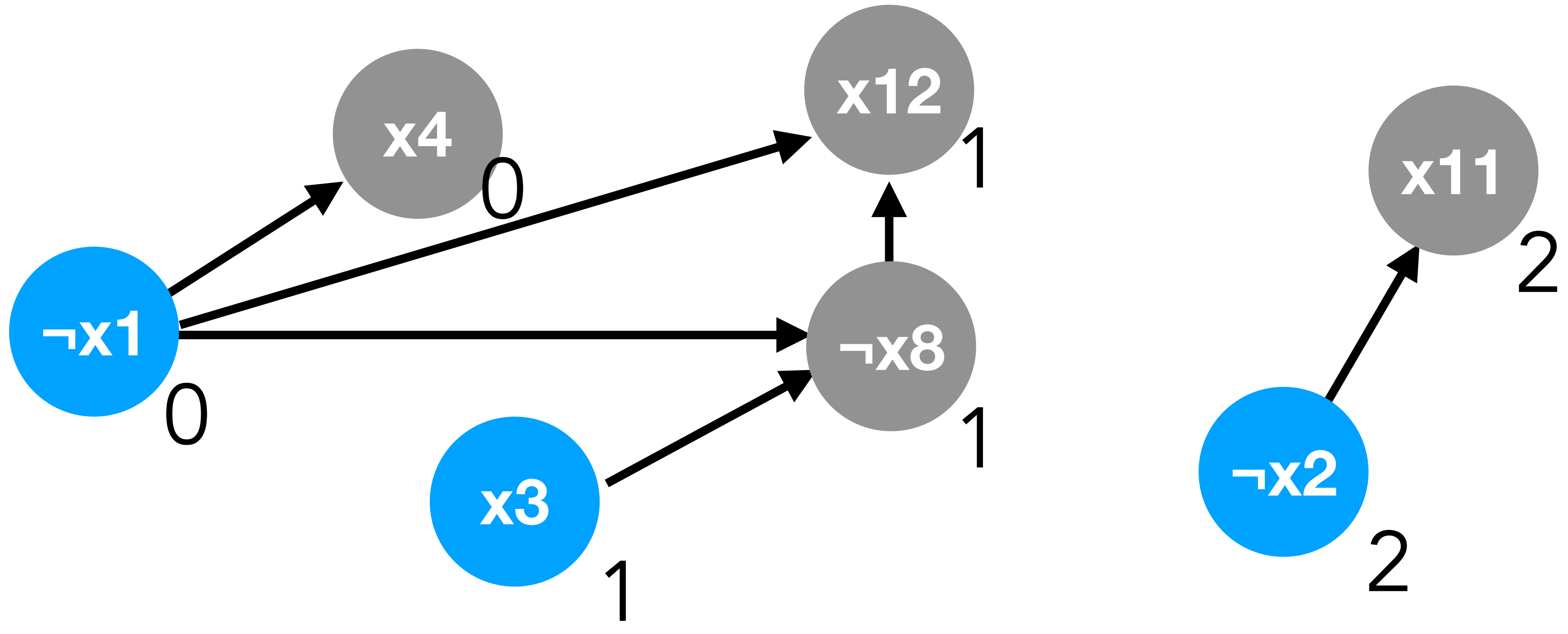
$x_7 \vee x_8 \vee \neg x_{10}$

$x_7 \vee x_{10} \vee \neg x_{12}$

We're now forced to decide x11



Implication Graph



Step 8 — Guess yet Again!

$x_1 \lor x_4$

$x_1 \lor \neg x_3 \lor \neg x_8$

$x_1 \lor \neg x_8 \lor x_{12}$

$x_2 \lor x_{11}$

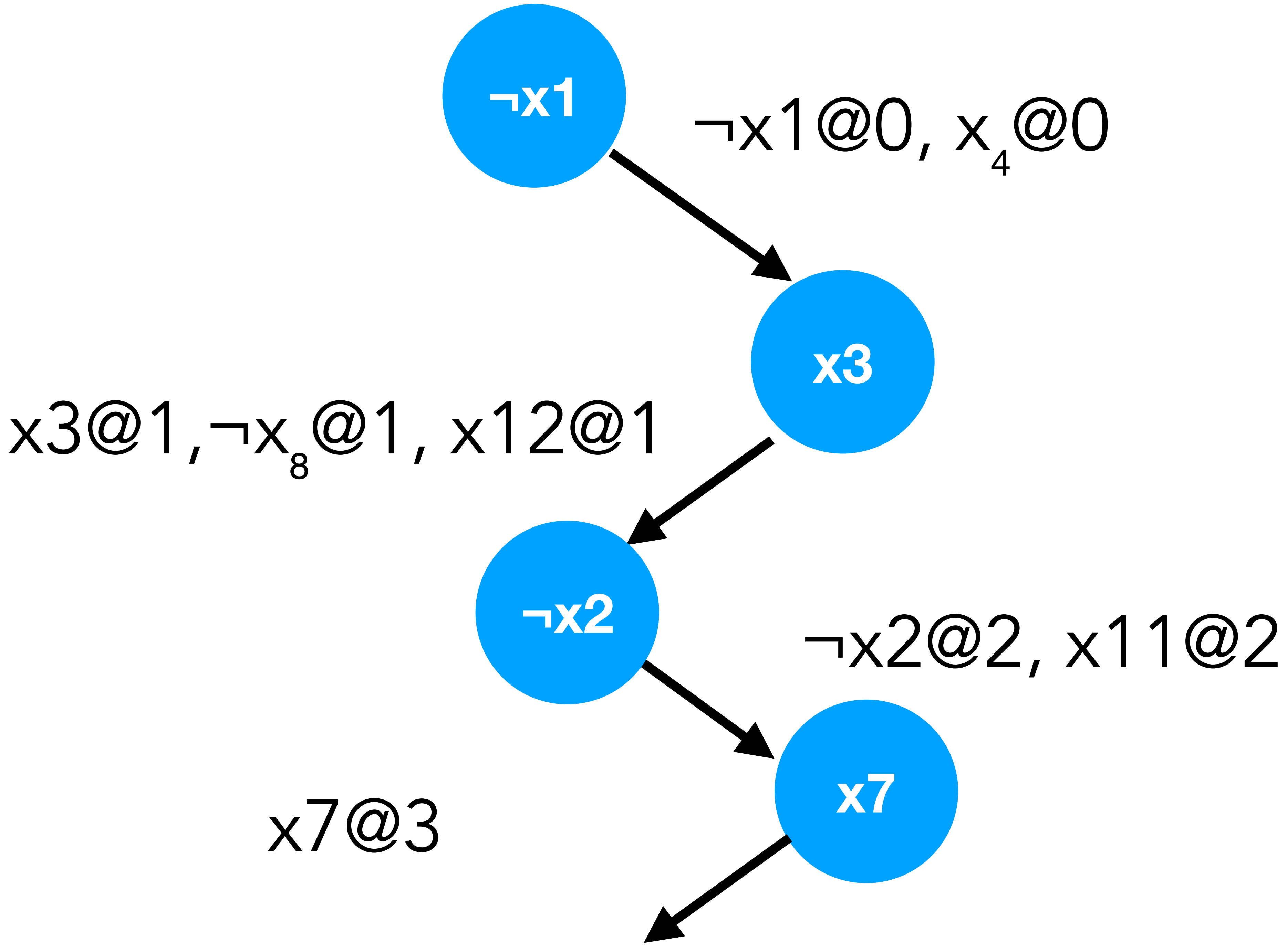
$\neg x_7 \lor \neg x_3 \lor x_9$

$\neg x_7 \lor x_8 \lor \neg x_9$

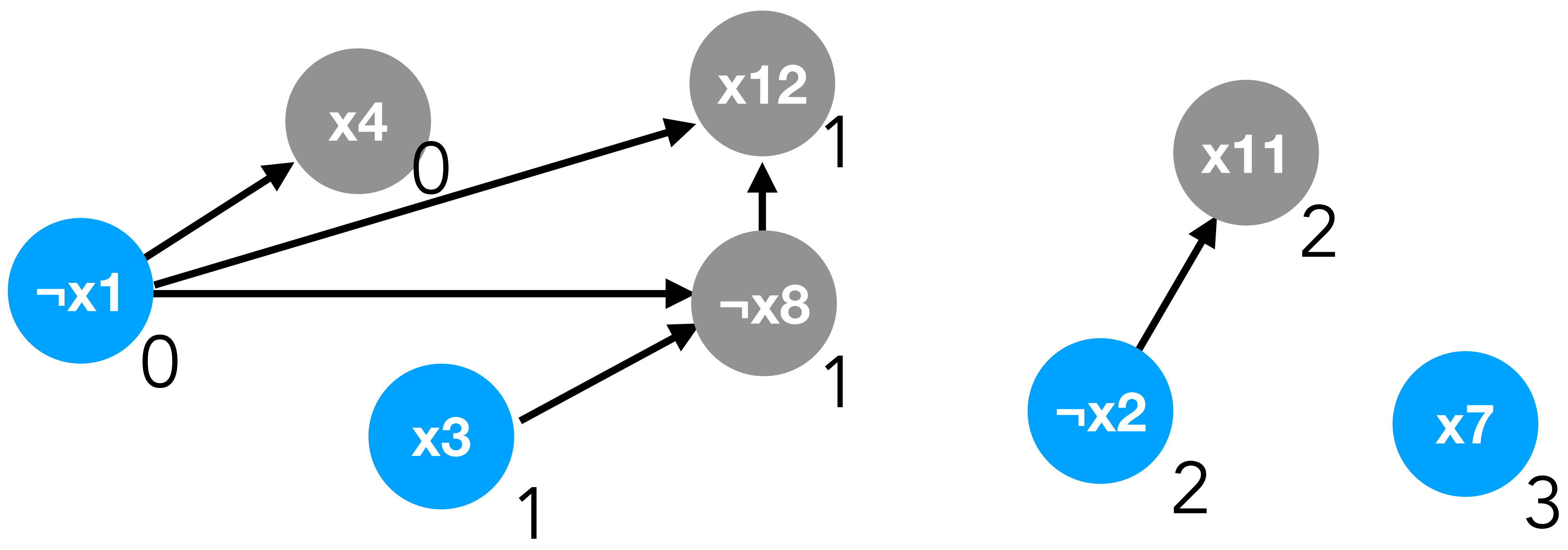
$x_7 \lor x_8 \lor \neg x_{10}$

$x_7 \lor x_{10} \lor \neg x_{12}$

Still no answer, let's try x7



Implication Graph



Step 9 — BCP & Conflict

$x_1 \lor x_4$

$x_1 \lor \neg x_3 \lor \neg x_8$

$x_1 \lor \neg x_8 \lor x_{12}$

$x_2 \lor x_{11}$

$\neg x_7 \lor \neg x_3 \lor x_9$

$\neg x_7 \lor x_8 \lor \neg x_9$

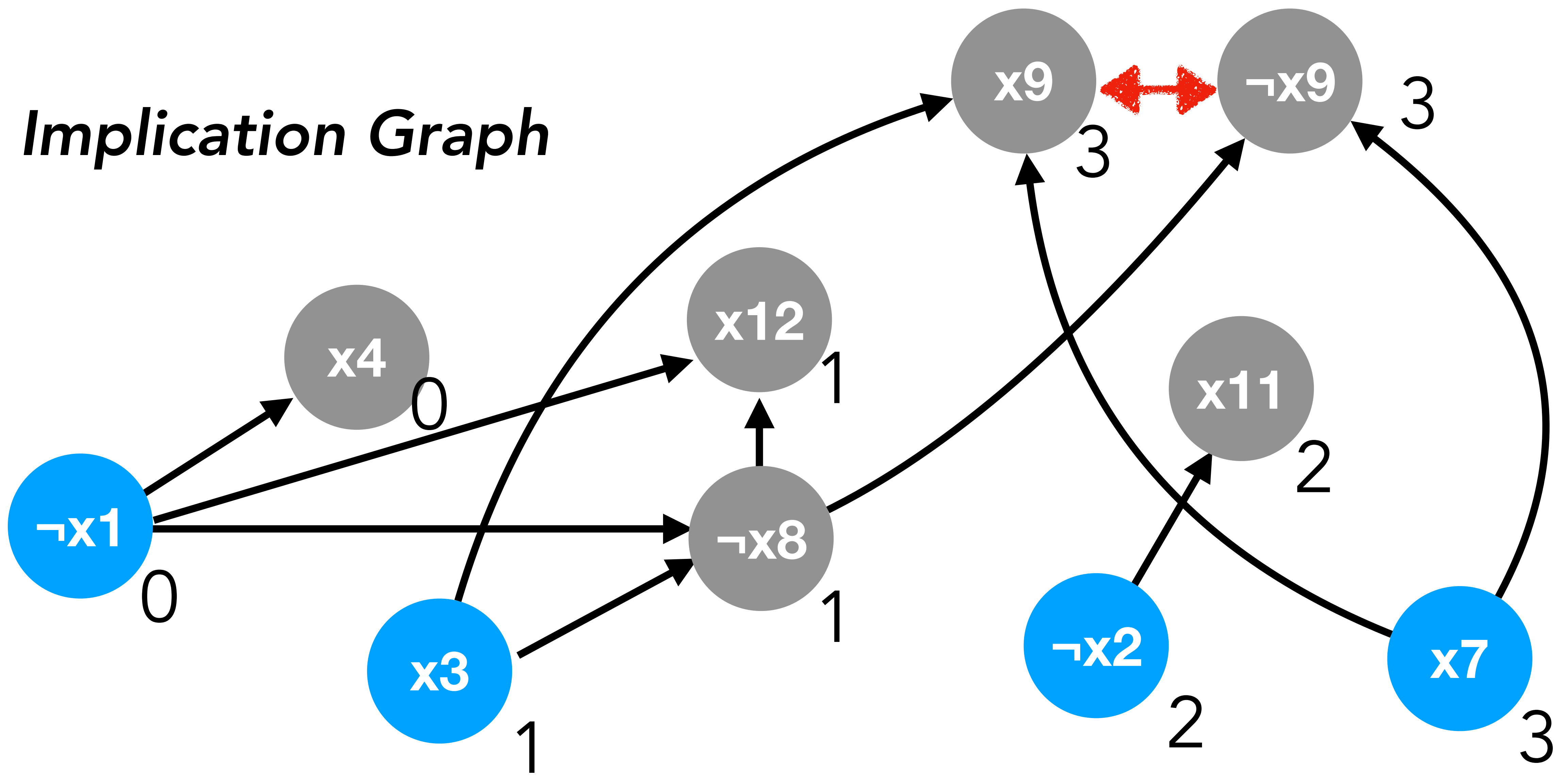
$x_7 \lor x_8 \lor \neg x_{10}$

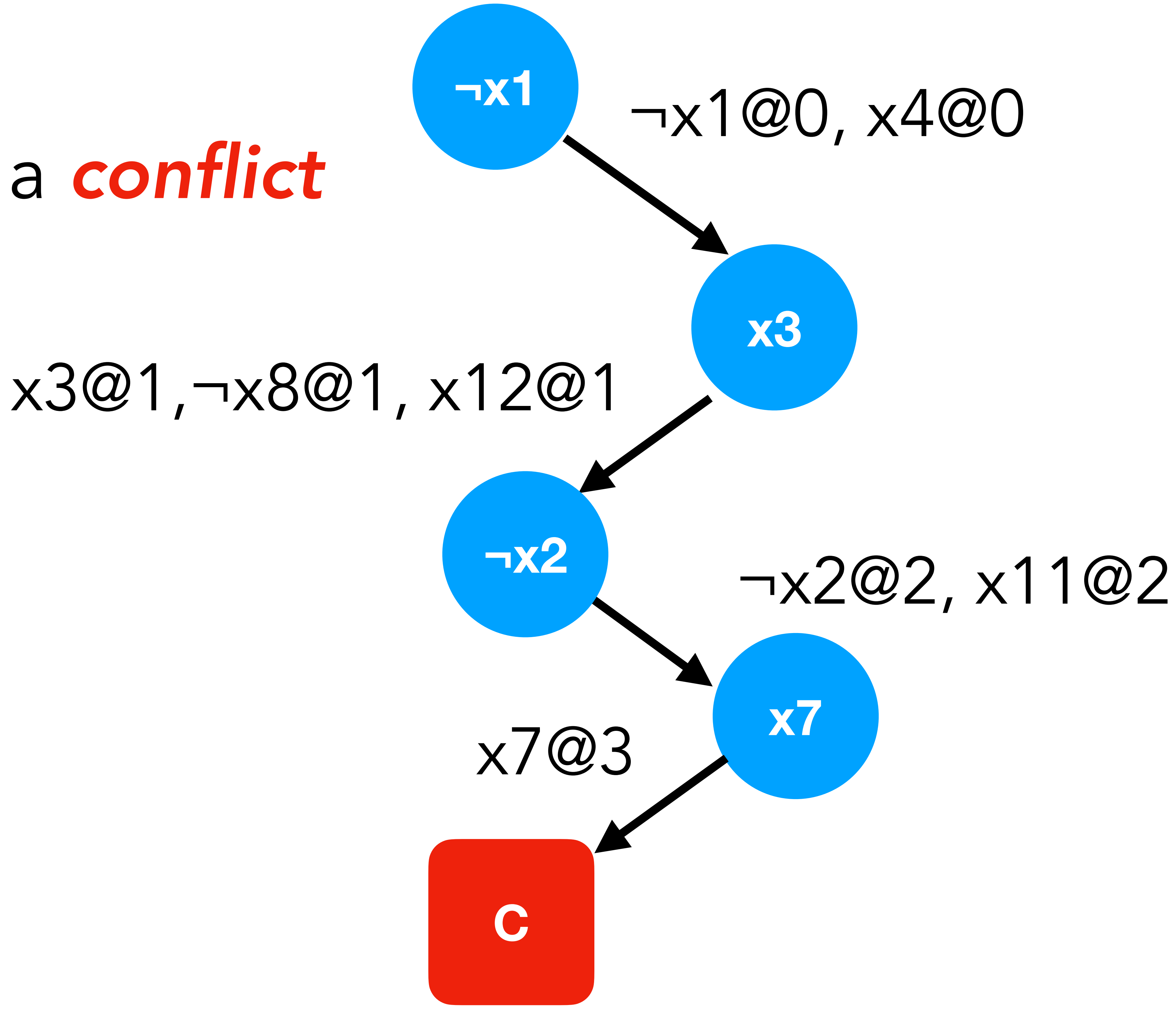
$x_7 \lor x_{10} \lor \neg x_{12}$

After deciding x7, we apply BCP

These two clauses yield a ***conflict***

Implication Graph



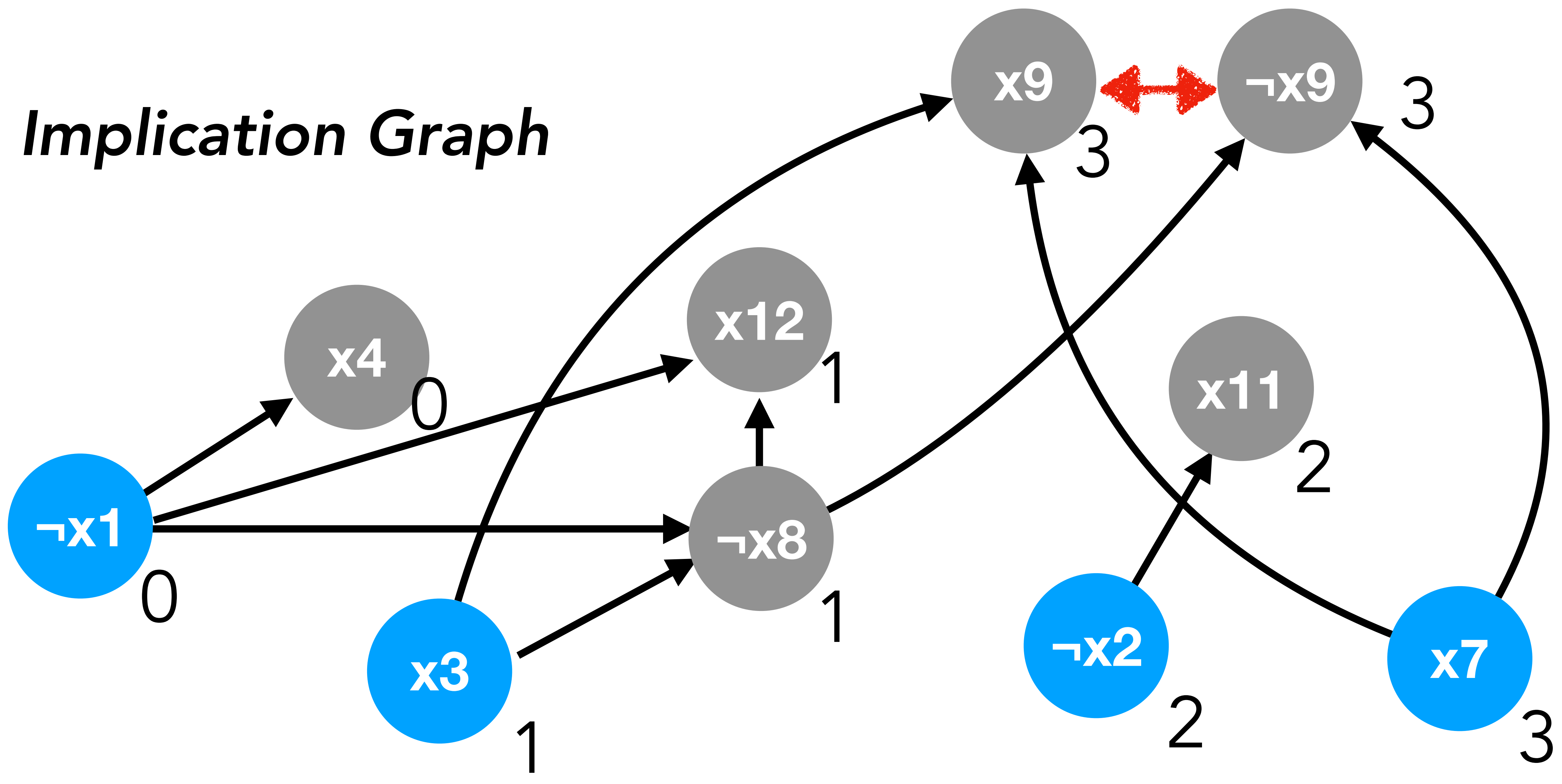


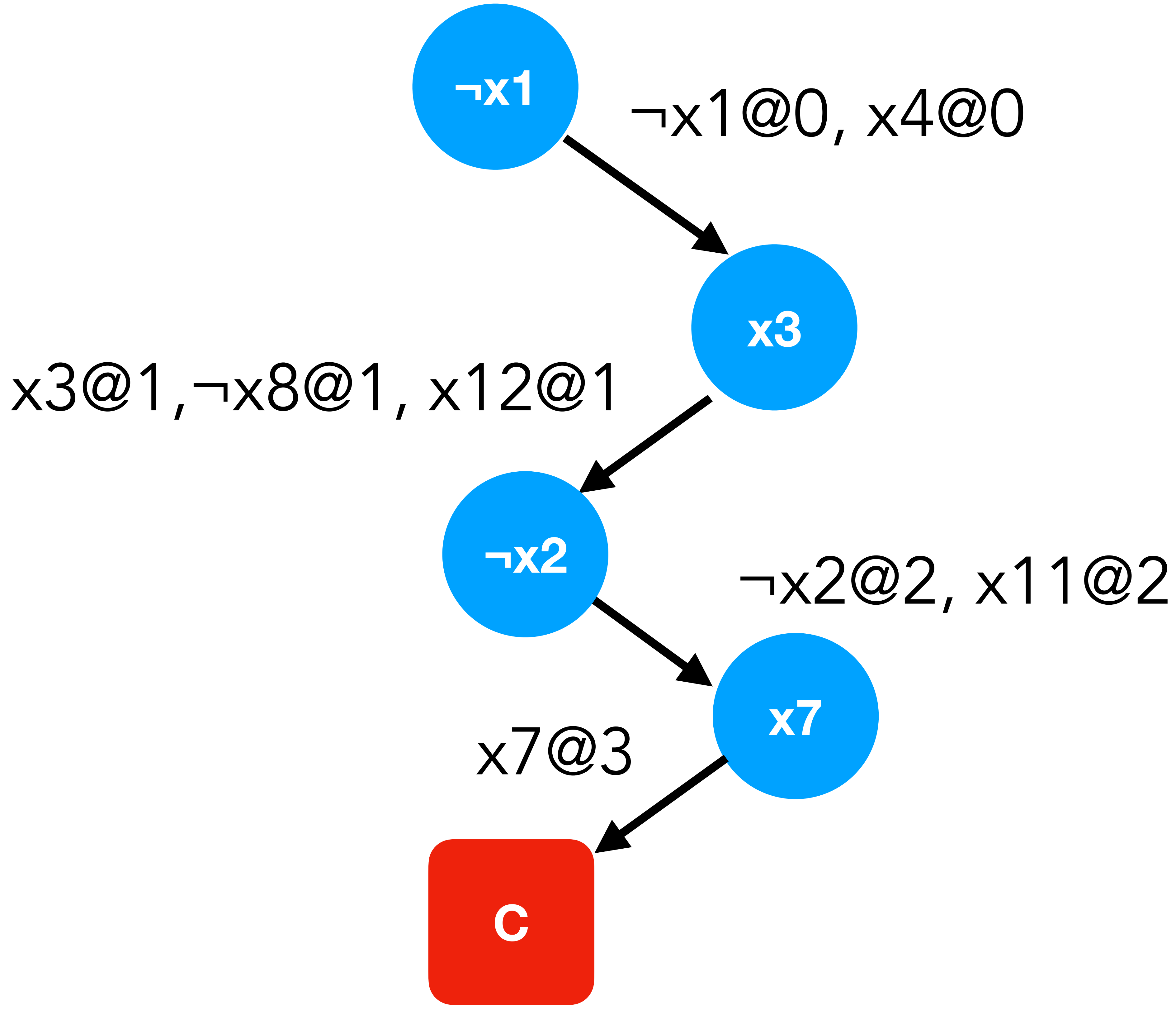
Step 10 — Analyze Conflict

$x_1 \vee x_4$
$x_1 \vee \neg x_3 \vee \neg x_8$
$x_1 \vee \neg x_8 \vee x_{12}$
$x_2 \vee x_{11}$
$\neg x_7 \vee \neg x_3 \vee x_9$
$\neg x_7 \vee x_8 \vee \neg x_9$
$x_7 \vee x_8 \vee \neg x_{10}$
$x_7 \vee x_{10} \vee \neg x_{12}$

We're at a conflict, we need to (a) decide on new "learned" clause and (b) decide where to backjump

Implication Graph





Step 10 — Analyze Conflict

$x_1 \vee x_4$

$x_1 \vee \neg x_3 \vee \neg x_8$

$x_1 \vee \neg x_8 \vee x_{12}$

$x_2 \vee x_{11}$

$\neg x_7 \vee \neg x_3 \vee x_9$

$\neg x_7 \vee x_8 \vee \neg x_9$

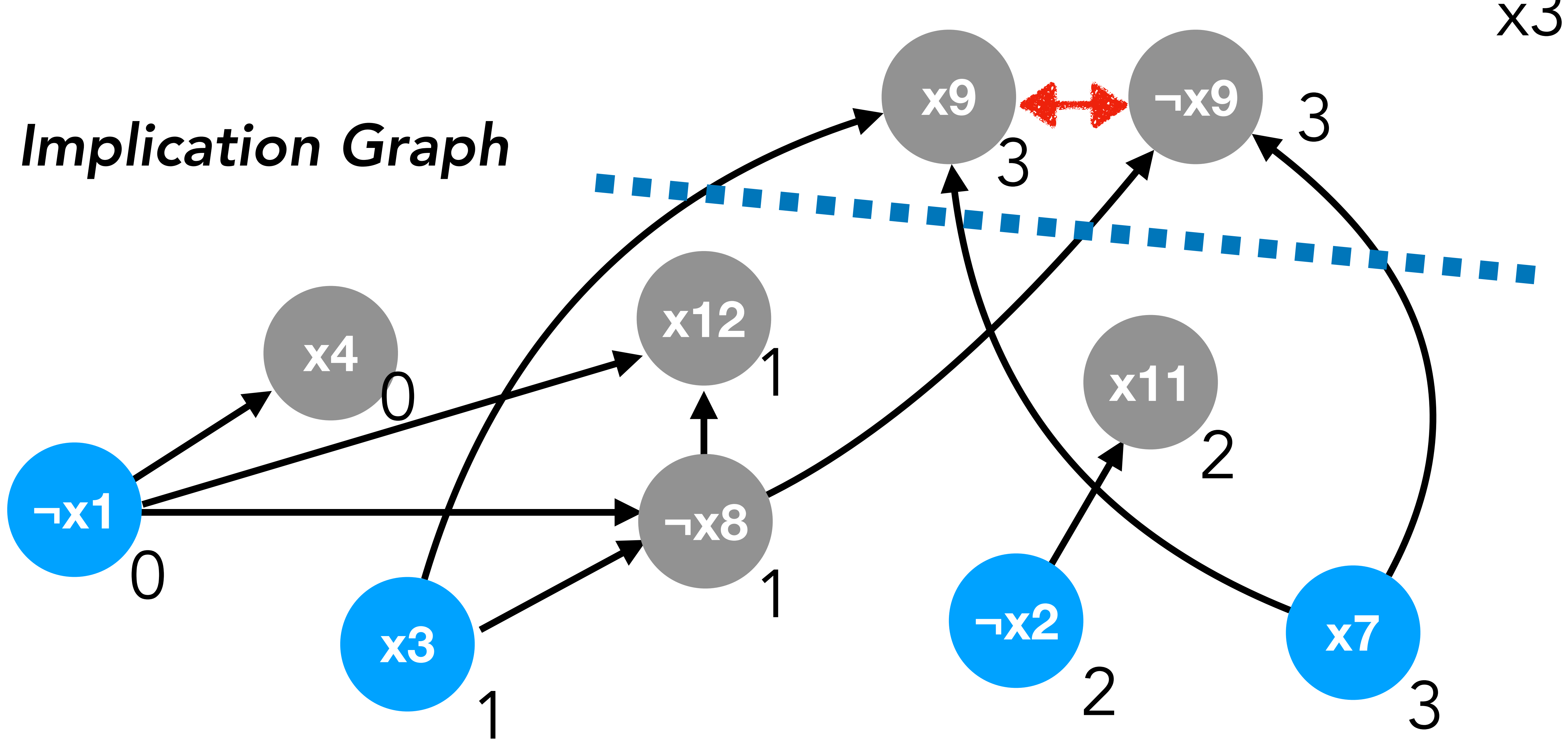
$x_7 \vee x_8 \vee \neg x_{10}$

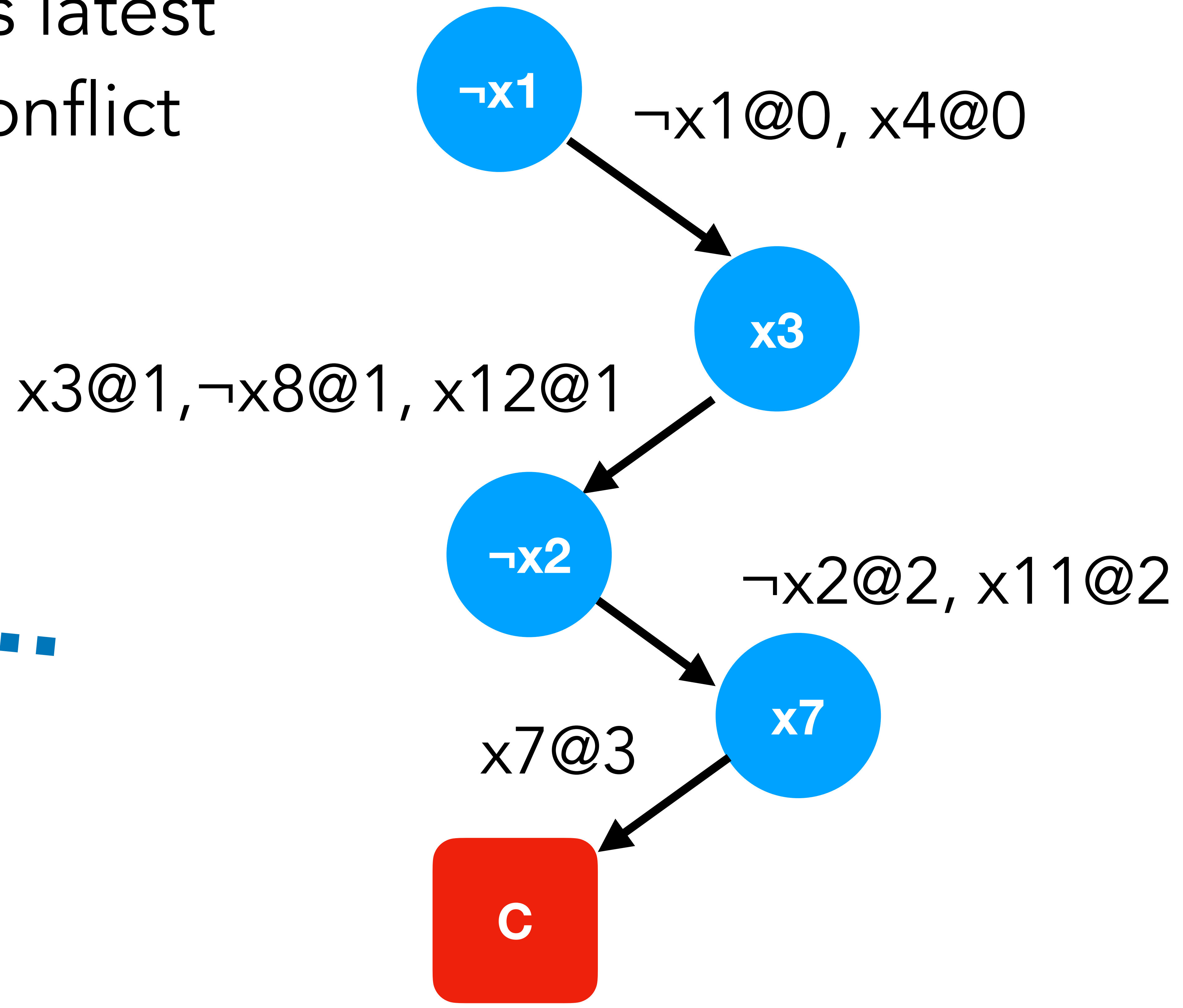
$x_7 \vee x_{10} \vee \neg x_{12}$

Idea: “cut out” the conflict

Identify cut in implication graph which separates latest decision node and conflict

Implication Graph





Step 10 — Analyze Conflict

$x_1 \vee x_4$

$x_1 \vee \neg x_3 \vee \neg x_8$

$x_1 \vee \neg x_8 \vee x_{12}$

$x_2 \vee x_{11}$

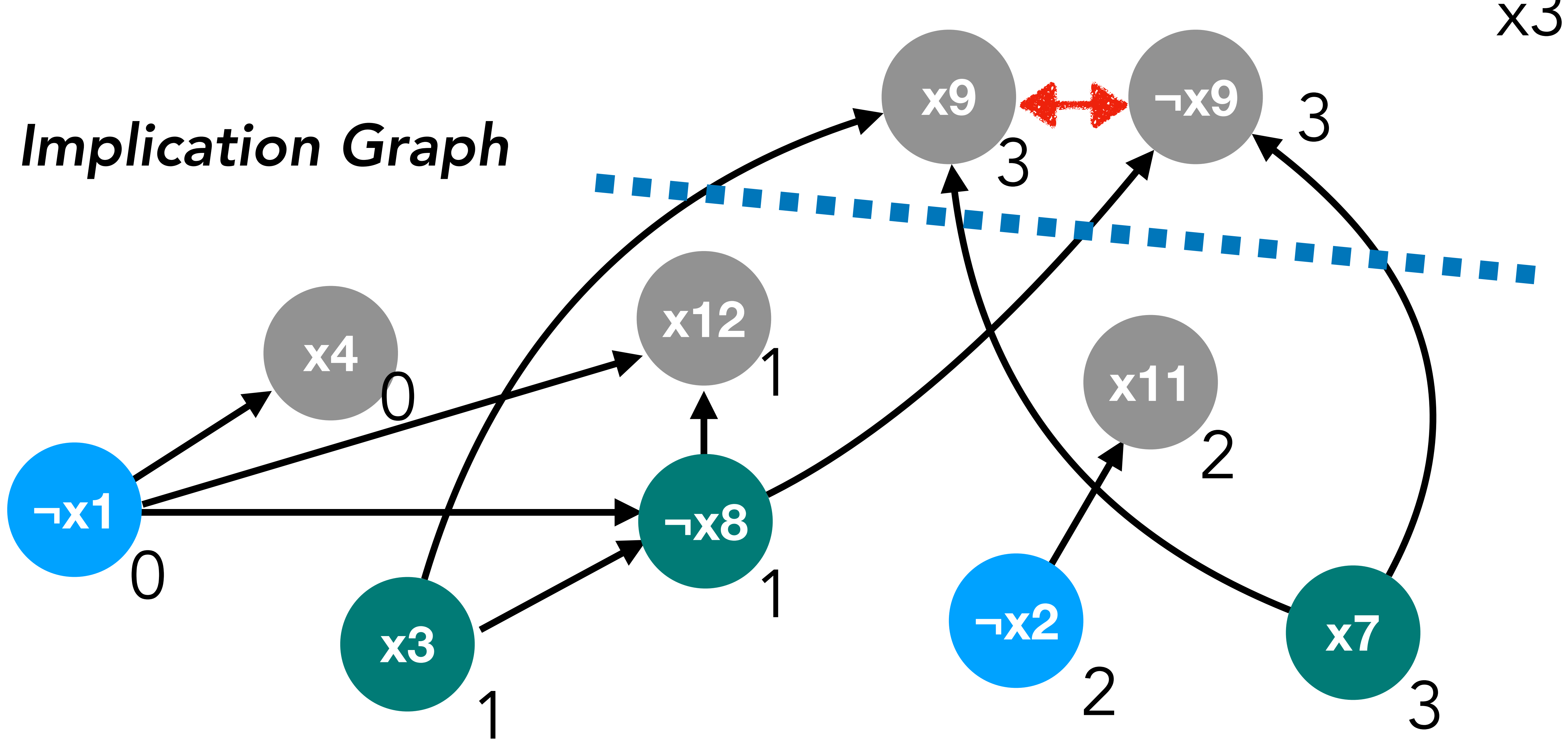
$\neg x_7 \vee \neg x_3 \vee x_9$

$\neg x_7 \vee x_8 \vee \neg x_9$

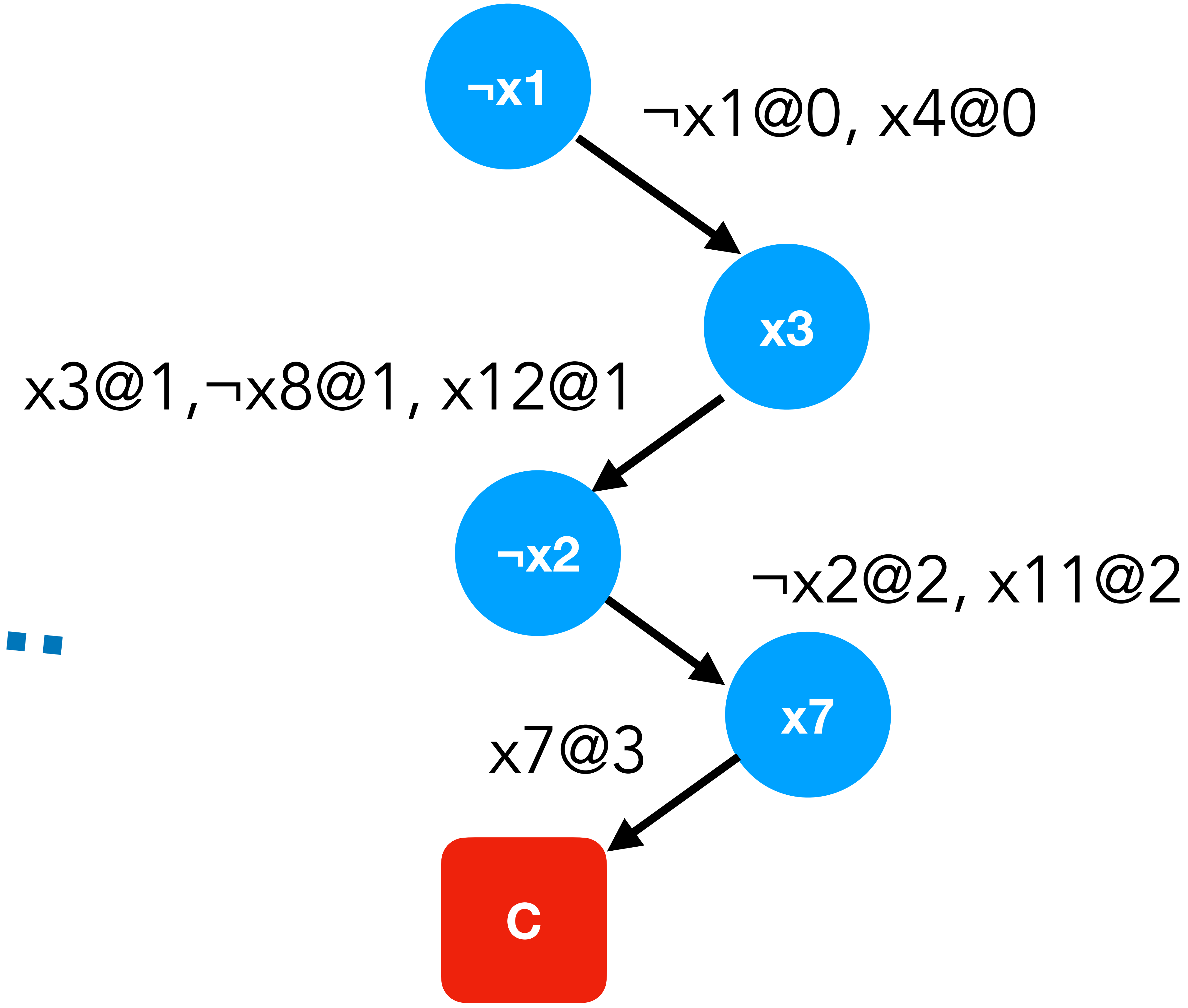
$x_7 \vee x_8 \vee \neg x_{10}$

$x_7 \vee x_{10} \vee \neg x_{12}$

Implication Graph



The “reason” is the incoming nodes sitting along the boundary



Together, these nodes form a ***sufficient condition*** for the conflict

Step 10 — Analyze Conflict

$x_1 \lor x_4$

$x_1 \lor \neg x_3 \lor \neg x_8$

$x_1 \lor \neg x_8 \lor x_{12}$

$x_2 \lor x_{11}$

$\neg x_7 \lor \neg x_3 \lor x_9$

$\neg x_7 \lor x_8 \lor \neg x_9$

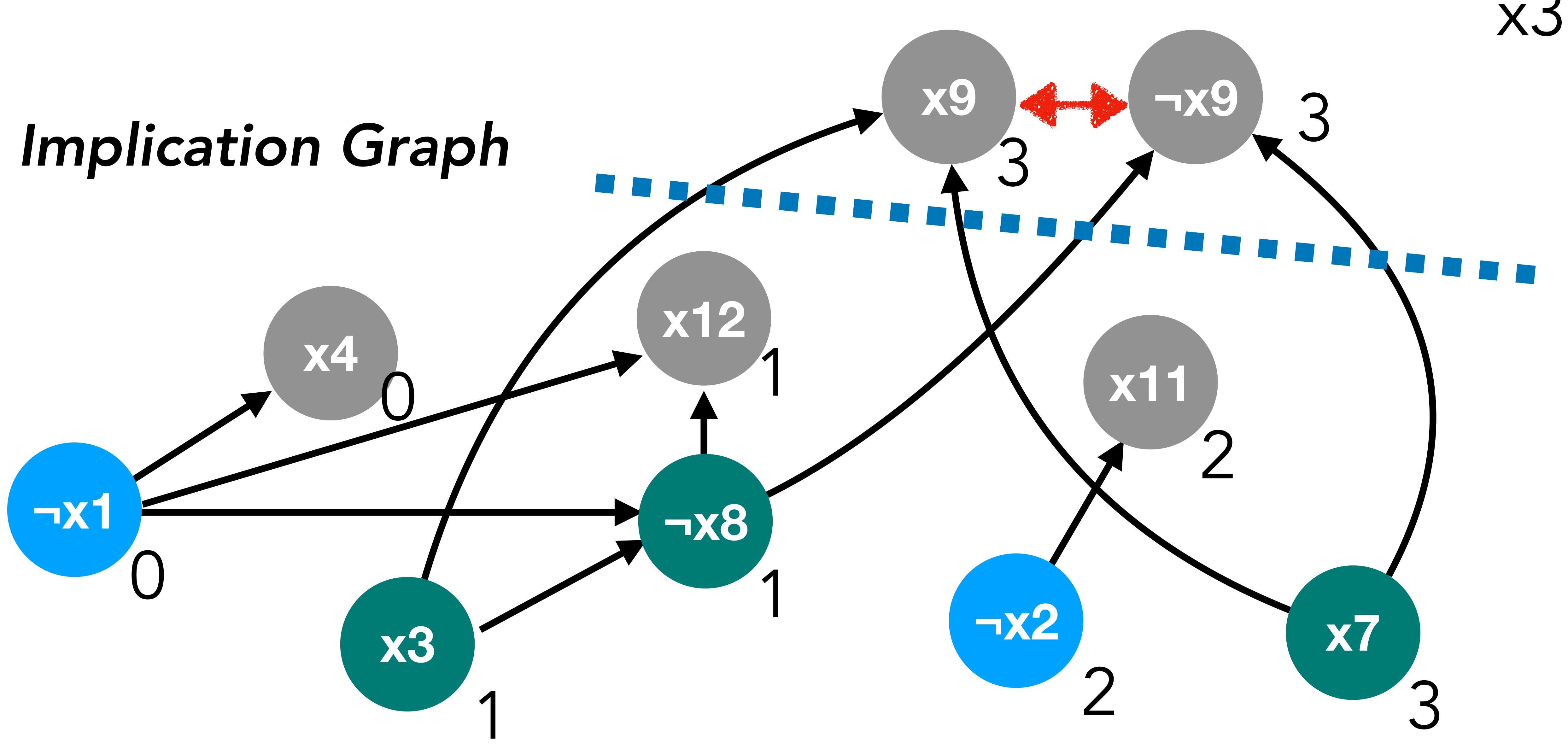
$x_7 \lor x_8 \lor \neg x_{10}$

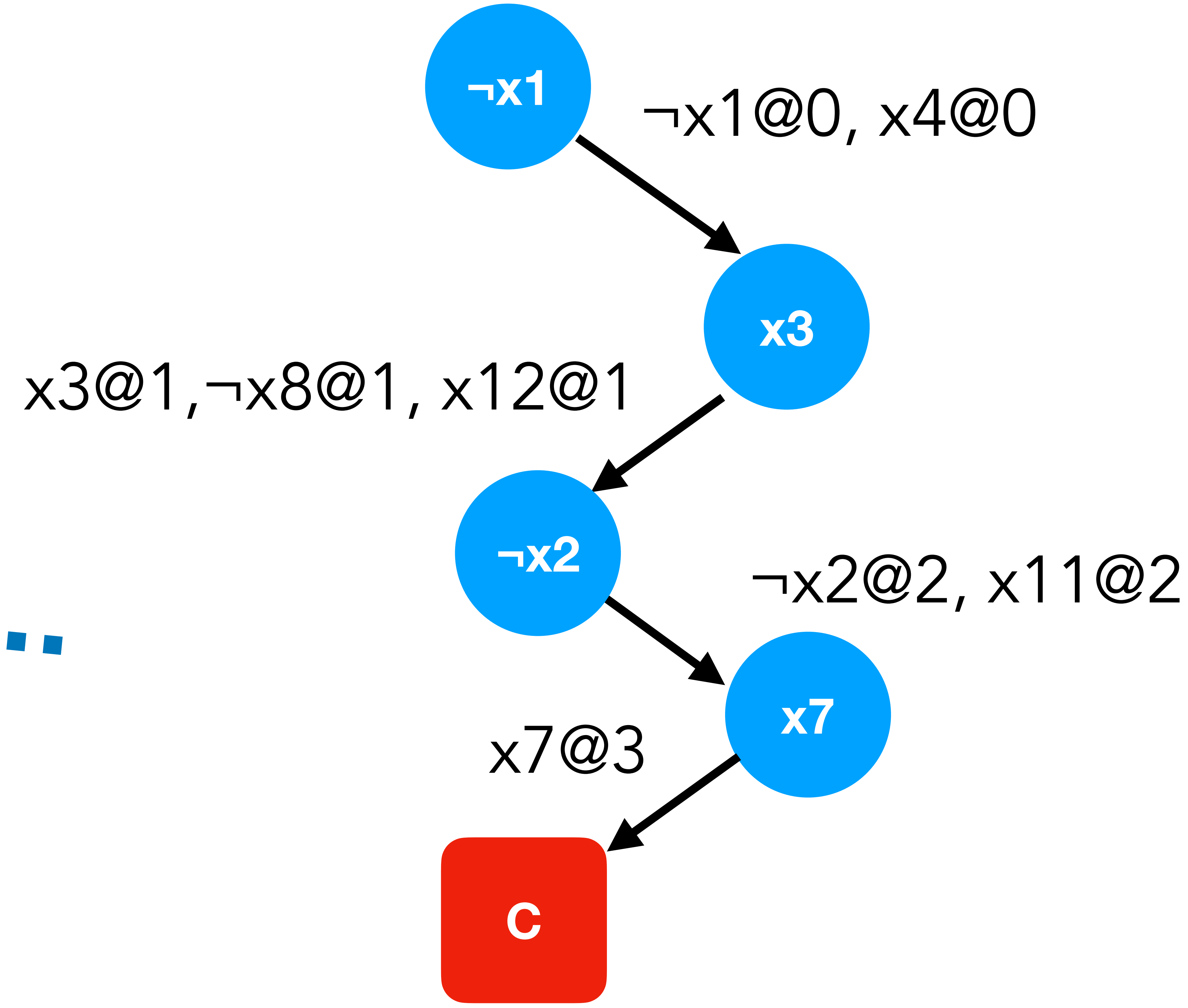
$x_7 \lor x_{10} \lor \neg x_{12}$

Thus, to "explain" the conflict we can assert $\neg(x_3 \land \neg x_8 \land x_7)$

i.e., $\neg x_3 \lor x_8 \lor \neg x_7$

Implication Graph





Step 11 — Backtrack

$x_1 \lor x_4$

$x_1 \lor \neg x_3 \lor \neg x_8$

$x_1 \lor \neg x_8 \lor x_{12}$

$x_2 \lor x_{11}$

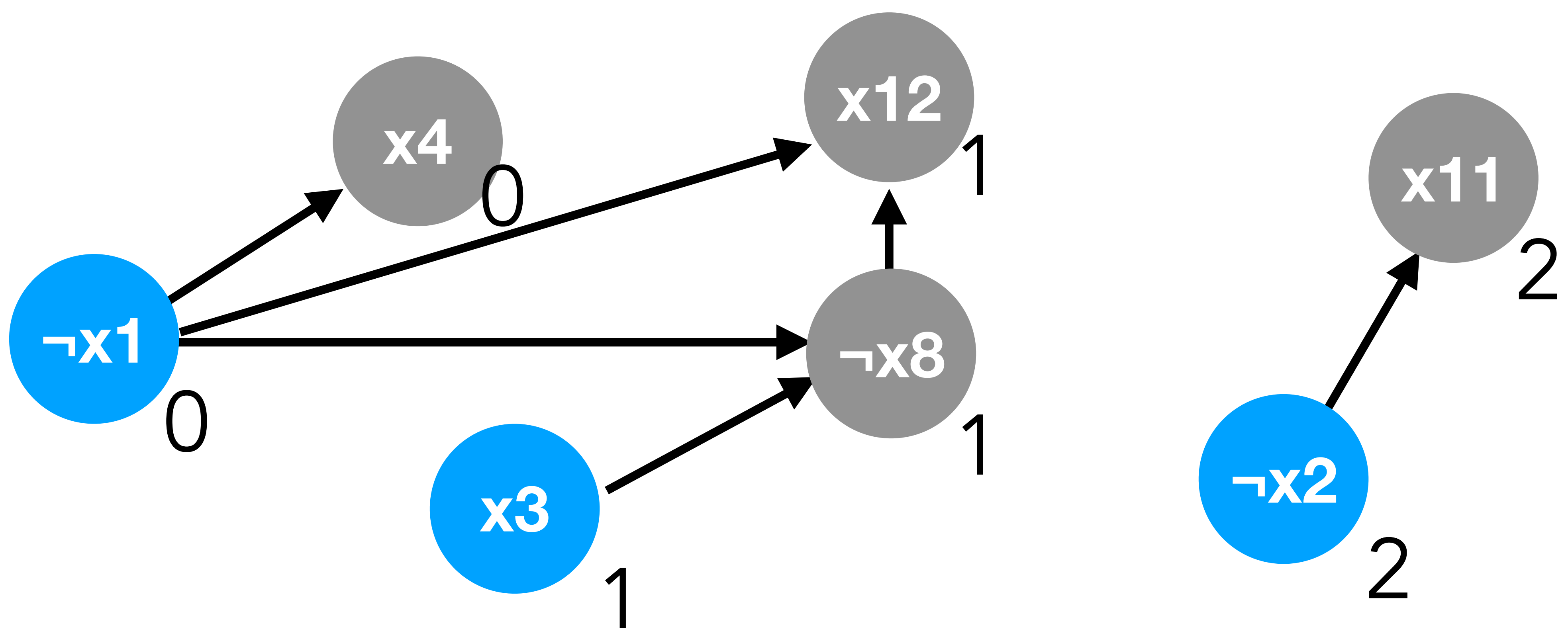
$\neg x_7 \lor \neg x_3 \lor x_9$

$\neg x_7 \lor x_8 \lor \neg x_9$

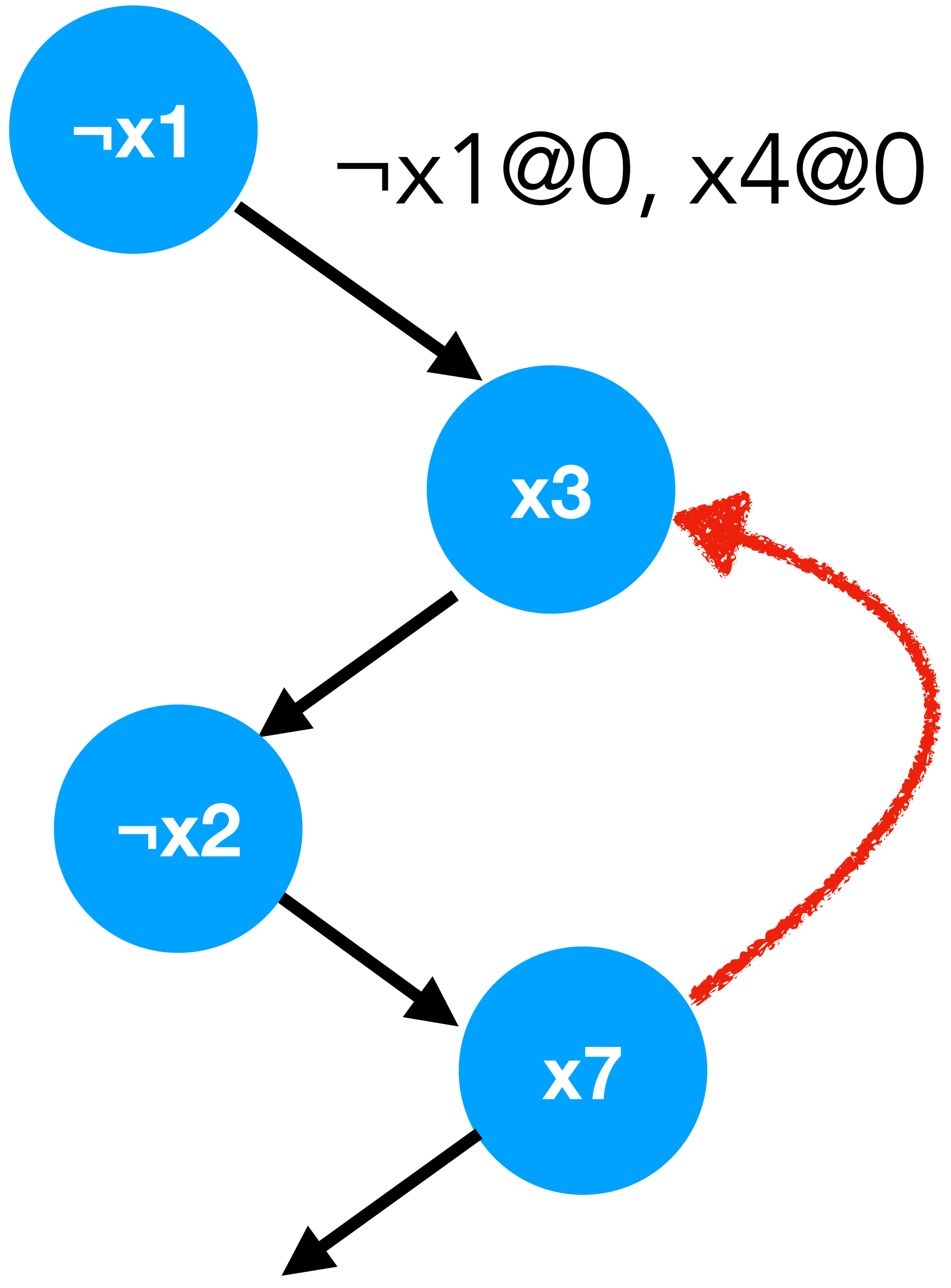
$x_7 \lor x_8 \lor \neg x_{10}$

$x_7 \lor x_{10} \lor \neg x_{12}$

Implication Graph



Now we “backtrack” to x3



Non-Chronological Backtracking

In DPLL, we backtrack “one level.”
In CDCL, we backtrack to the *second most recent decision level in the conflict clause*. Or, equivalently, backtrack to the highest decision level in the conflict clause other than the most recent decision level.