Tail Calls and Tail Recursion

CIS352 — Spring 2023

Kris Micinski

```
((lambda (x) x) ((lambda (y) y) 5))

((lambda (x) x) 5)

5
```

Calculating factorial in Racket

```
(define (factorial n)
  (if (= n 0)
      1
      (* n (factorial (sub1 n)))))
```

Calculating factorial in Racket

```
(define (factorial n)
  (if (= n 0)
      1
      (* n (factorial (sub1 n)))))
```

Defines **base case**

Calculating factorial in Racket

```
(define (factorial n)
  (if (= n 0)
      1
      (* n (factorial (sub1 n)))))
```

and **inductive / recursive** case

Calculating factorial in Racket

```
(define (factorial n)
  (if (= n 0)
      1
      (* n (factorial (sub1 n)))))
```

We can think of recursion as “substitution”

```
> (factorial 2)
```

```
(define (factorial n)
  (if (= n 0)
      1
      (* n (factorial (sub1 n)))))
```

We can think of recursion as “substitution”

```
> (factorial 2)
= (if (= 2 0)
      1
      (* 2 (factorial (sub1 2))))
```

Copy defn, substitute for argument `n`

```
(define (factorial n)
  (if (= n 0)
      1
      (* n (factorial (sub1 n)))))
```

We can think of recursion as “substitution”

```
> (factorial 2)
= (if (= 2 0)
      1
      (* 2 (factorial (sub1 2))))
= (if #f 1 (* 2 (factorial (sub1 2))))
= (* 2 (factorial (sub1 2)))
= (* 2 (factorial 1))
= (* 2 (if …))
```

```
…
= (* 2 (if (= 2 0)
         1
         (* n (factorial (sub1 2)))))
= (* 2 (factorial 1))
= …
= (* 2 (* 1 1))
= (* 2 1)
= 2
```

Notice we're building a big stack of calls to *

Tail Calls

- Unlike calls in general, ***tail calls*** do not affect the stack:
  - Tail calls *do not grow* (or shrink) the stack.
    - They are more like a goto/jump than a normal call.

Tail Position

- A subexpression is in ***tail position*** if it's:
  - The last subexpression to run, whose return value is also the value for its parent expression
  - In `(let ([x rhs]) body)`; <u>body</u> is in *tail position*…
  - In `(if grd thn els)`; <u>thn</u> & <u>els</u> are in *tail position*…

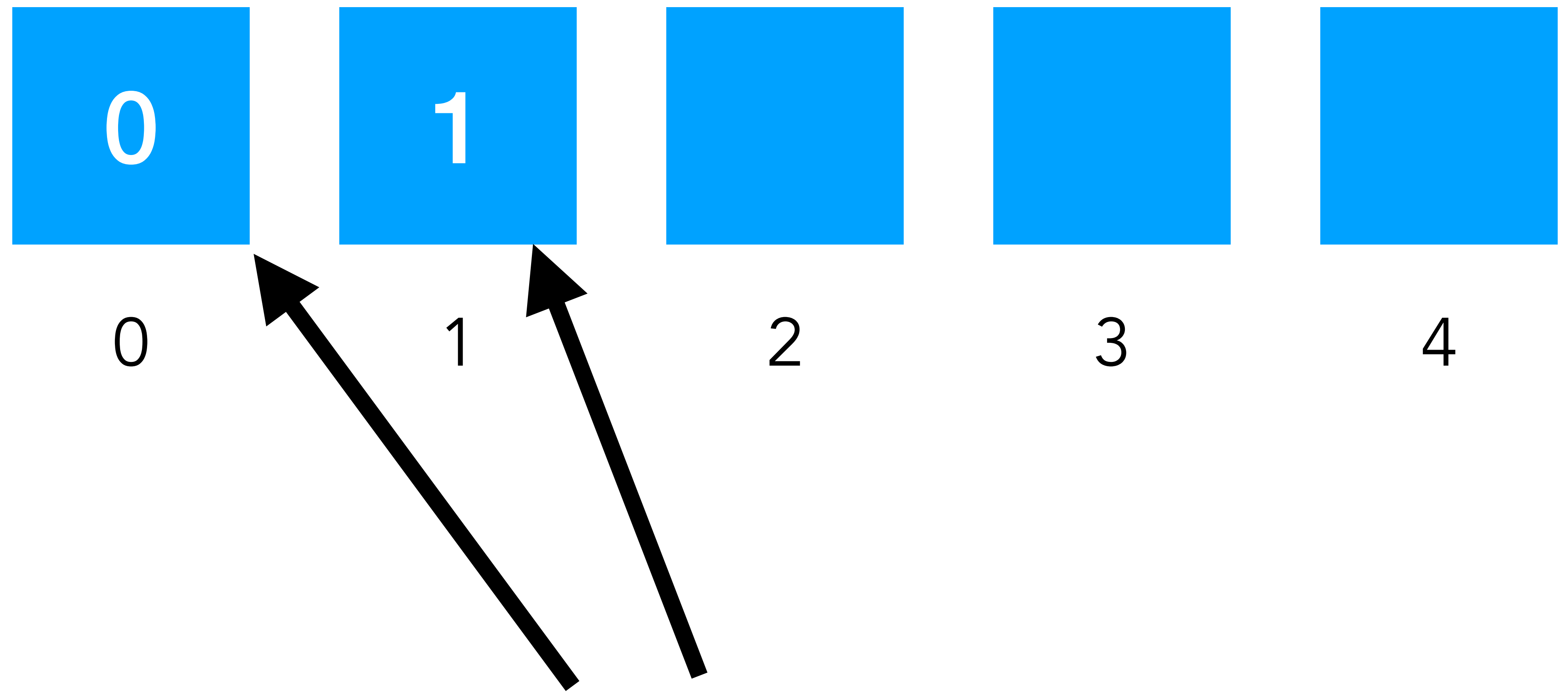
Tail Recursion

- A function is ***tail recursive*** if all recursive calls in tail position
- Tail-recursive functions are analogous to loops in imperative langs

Tail calls / tail recursion

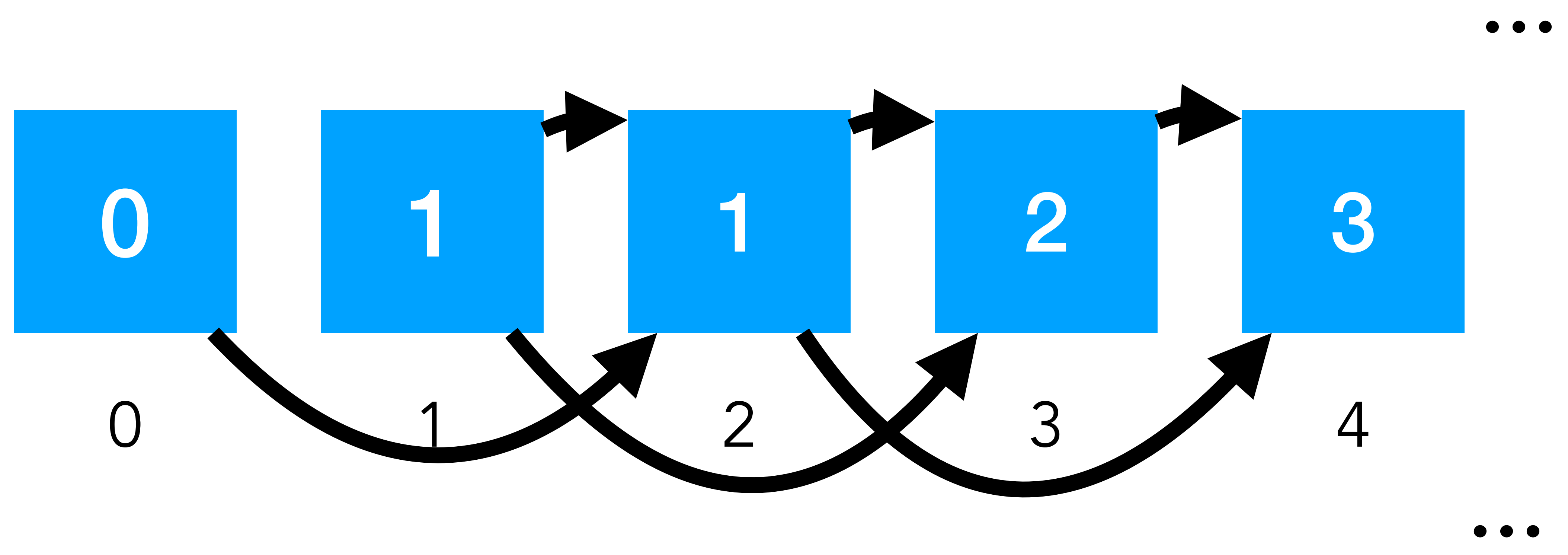
- Unlike calls in general, ***tail calls*** do not affect the stack:
  - Tail calls *do not grow* (or shrink) the stack.
    - They are more like a goto/jump than a normal call.
- A function is ***tail recursive*** if all recursive calls in tail position
- Tail-recursive functions are analogous to loops in imperative langs

Instead, use ***dynamic programming:***
design a recursive solution top-down, but implement as a bottom-up algorithm!

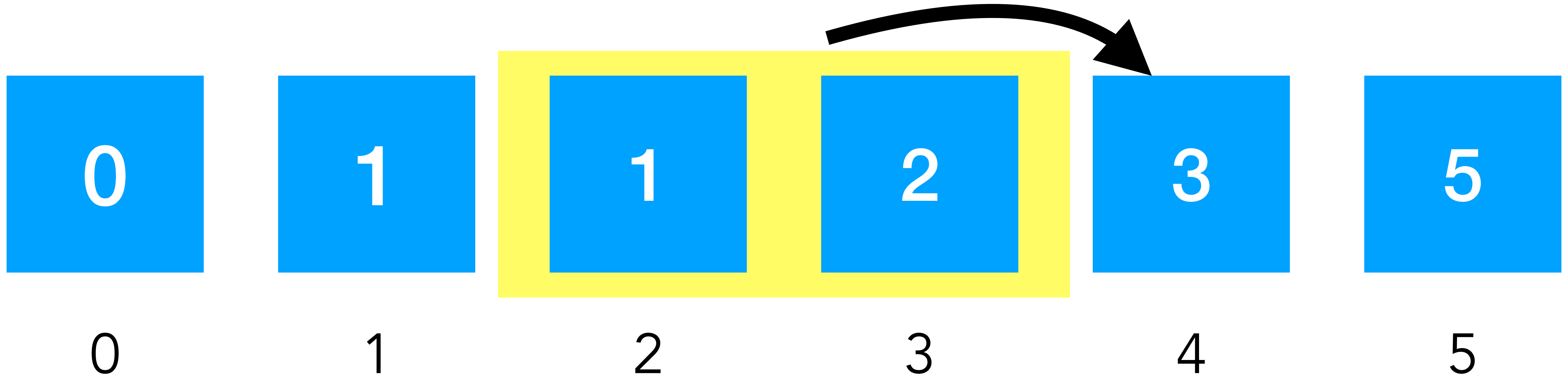


Start with first two, then build up

Instead, use ***dynamic programming:***
design a recursive solution top-down, but implement as a bottom-up algorithm!



Key idea: only need to look at **two most recent** numbers



Accumulate via arguments

```
(define (fib-h i n0 n1)
  (if (= i 0)
      n0
      (fib-h (- i 1) n1 (+ n0 n1))))

(define (fib n) (fib-h n 0 1))
```

Exercise

```
(define (fib-h i n0 n1)
  (if (= i 0)
      n0
      (fib-h (- i 1) n1 (+ n0 n1))))

(define (fib n) (fib-h n 0 1))
```

Question: what is the runtime complexity of `fib`?

Exercise

```
(define (fib-h i n0 n1)
  (if (= i 0)
      n0
      (fib-h (- i 1) n1 (+ n0 n1))))

(define (fib n) (fib-h n 0 1))
```

Answer: O(n), fib-helper runs from `n` to `0`

Consider how `fib-h` executes

```
(define (fib-h i n0 n1)
  (if (= i 0)
      n0
      (fib-h (- i 1) n1 (+ n0 n1))))

(define (fib n) (fib-h n 0 1))
```

```
(fib-helper 3 0 1)
= (if (= 3 0) 0 (fib-h (- 3 1) 1 (+ 0 1)))
= …
= (fib-h 2 1 1)
= (if (= 2 0) 1 (fib-h (- 2 1) 1 (+ 1 1)))
= …
= (fib-h 1 1 2)
```

Notice that we don’t get the “stacking” behavior:
recursive calls don’t grow the stack

This is because `fib-h` is **tail recursive**

```
(define (fib-h i n0 n1)
  (if (= i 0)
      n0
      (fib-h (- i 1) n1 (+ n0 n1))))

(define (fib n) (fib-h n 0 1))
```

Intuitively: a callsite is in **tail-position** if it is the **last thing** a function will do before exiting

(We call these **tail calls**)

This is because `fib-h` is **tail recursive**

Both of these are tail calls

```
(define (fib-h i n0 n1)
  (if (= i 0)
      n0
      (fib-h (- i 1) n1 (+ n0 n1))))

(define (fib n) (fib-h n 0 1))
```

Intuitively: a callsite is in **tail-position** if it is the **last thing** a function will do before exiting

(We call these **tail calls**)

Tail calls / tail recursion

- Unlike calls in general, ***tail calls*** do not affect the stack:
  - Tail calls *do not grow* (or shrink) the stack.
    - They are more like a goto/jump than a normal call.
- A subexpression is in ***tail position*** if it's the last subexpression to run, whose return value is also the value for its parent expression:
  - In `(let ([x rhs]) body)`; body is in *tail position*…
  - In `(if grd thn els)`; thn *&* els are in *tail position*…
- A function is ***tail recursive*** if all recursive calls in tail position
- Tail-recursive functions are analogous to loops in imperative langs

Exercise

Which of the following is tail recursive?

```
(define (length-0 l)
  (if (null? l)
      0
      (+ 1 (length-0 (cdr l)))))

(define (length-1 l n)
  (if (null? l)
      n
      (length-1 (cdr l) (+ n 1))))
```

Exercise

Answer

```
(define (length-0 l)
  (if (null? l)
      0
      (+ 1 (length-0 (cdr l)))))
```

Not tail recursive

Adds (+ 1 _) operation to stack

```
(define (length-1 l n)
  (if (null? l)
      n
      (length-1 (cdr l) (+ n 1))))
```

Is tail recursive!

Call to length-1 in tail position