The Lambda Calculus

CIS 352 — Spring 2020

λ

The Lambda Calculus

- A system for calculating based entirely on computing with functions.
- Developed as a foundation for mathematics (originally used to model the natural numbers) by ***Alonzo Church*** in 1936.
- Church's thesis: *"Every effectively calculable function (effectively decidable predicate) is general recursive"*, i.e., can be computed using the λ-calculus. Used to show there exist unsolvable problems.
- One of the simplest Turing-equivalent languages!
    - Church, with his student Alan Turing, proved the equivalent expressiveness of Turing machines and the λ-calculus (called the ***Church-Turing thesis***).
- Still makes up the heart of all functional programming languages!

The Lambda Calculus

lambdas are just anonymous functions!

$e \in \textbf{Exp} ::= (\lambda\ (x)\ e)$	λ-abstraction
$\quad\mid (e\ e)$	function application
$\quad\mid x$	variable reference

$$x \in \textbf{Var} ::= \langle \textbf{variables} \rangle$$

Textual-reduction semantics

- One way of designing a formal semantics is as a relation over terms in the language—one that reduces the term textually.

- This is usually ***small-step***—each eval step must terminate (meaning there are no *premises above the line* in our rules of inference and no recursive use of the interpreter within a step.)

- Consider a small-step semantics for our arithmetic language:

$$a \in \textbf{AExp} \;\; ::= \;\; n \mid a + a \mid a - a \mid a \times a$$

$$n, m \in \textbf{Num} \;\; ::= \langle \textbf{integer constants} \rangle$$

Exercise

Which of the following are **AExp**s:

- 10
- 20.5
- 10 + 3
- 10 + 3 * 4^2
- 5 - 3 * 2 + 3 * 1

$$a \in \textbf{AExp} \ ::= \ n \mid a + a \mid a - a \mid a \times a$$

$$n, m \in \textbf{Num} \ ::= \langle \textbf{integer constants} \rangle$$

Exercise

Which of the following are **AExp**s:

- **10**
- 20.5 — No, not integer constant
- **10 + 3**
- 10 + 3 * 4^2 — Exponent not allowed
- **5 - 3 * 2 + 3 * 1**

$$a \in \textbf{AExp} \;::=\; n \mid a + a \mid a - a \mid a \times a$$

$$n, m \in \textbf{Num} \;::=\; \langle \textbf{integer constants} \rangle$$

Textual-reduction semantics

$$a \in \textbf{AExp} \;\; ::= \;\; n \mid a + a \mid a - a \mid a \times a$$
$$n, m \in \textbf{Num} \;\; ::= \langle \textbf{integer constants} \rangle$$

- Rules to reduce terms in this language match operations that have two numeric operands already and apply the operation, textually substituting a numeric value for the operation; e.g.:

$$\frac{}{a_0 \times a_1 \Rightarrow n_0 * n_1}$$ **where** a_0 is n_0 and a_1 is n_1

- For example: $2 * 3 + 4 * 5 \Rightarrow 2 * 3 + 20 \Rightarrow 6 + 20 \Rightarrow 26$

- Is there another way to evaluate 2*3 + 4*5 using similar rules?

The Lambda Calculus

lambdas are just anonymous functions!

$e \in \textbf{Exp} ::= (\lambda\ (x)\ e)$	λ-abstraction
$\quad\mid (e\ e)$	function application
$\quad\mid x$	variable reference

$$x \in \textbf{Var} ::= \langle\textbf{variables}\rangle$$

The Lambda Calculus

The lambda-calculus is the functional core of
Racket (as of other functional languages).

Just the following subset of Racket is Turing-equivalent!

$e \in \textbf{Exp} ::= (\lambda\ (x)\ e)$	`(lambda (x) e)`
$\quad \mid (e\ e)$	`(e0 e1)`
$\quad \mid x$	`x`

$$x \in \textbf{Var} ::= \langle \textbf{variables} \rangle$$

Lambda Abstraction

An expression, *abstracted* over all possible values for a formal parameter, in this case, x.

$$(\lambda\ (x)\ e)$$

Formal parameter

Function body

Application

When the first expression is evaluated to a value (in this language, all values are functions!) it may be invoked / applied on its argument.

$$(e\ \ e)$$

Expression in ***function position***

Expression in ***argument position***

Variables

Variables are only defined/assigned when a function is applied and its parameter bound to an argument.

$$x$$

Variable reference

```
((λ (f) (f (f (λ (x) x)))) (λ (x) x))
```

We define a rule for step-by-step evaluation called ***Beta-reduction***

$\downarrow \beta$

```
((λ (x) x) ((λ (x) x) (λ (x) x)))
```

$\downarrow \beta$

```
((λ (x) x) (λ (x) x))
```

$\downarrow \beta$

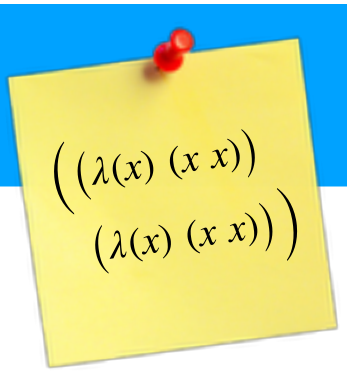
```
(λ (x) x)
```

Textual substitution. This says:
replace every x in E_0 with E_1.

$$\underbrace{((\lambda\ (x)\ E_0)\ E_1)}_{\text{redex}} \quad \to_\beta \quad E_0[x \leftarrow E_1]$$

(**red**ucible **ex**pression)

Example

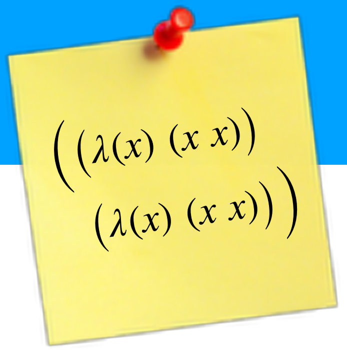



```
((λ (x) x) (λ (x) x))
```

$$\Big\downarrow \beta$$

```
x[x ← (λ (x) x)]
```

Example

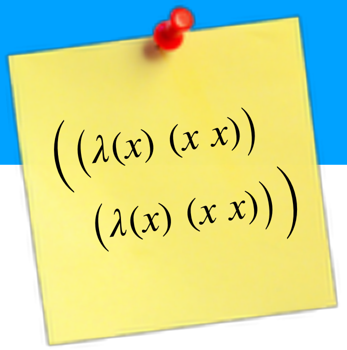


$$((\lambda\ (x)\ x)\ (\lambda\ (x)\ x))$$

$$\Big\downarrow \beta$$

$$(\lambda\ (x)\ x)$$

Example



Can you beta-reduce the following term more than once:

```
((λ (x) (x x)) (λ (x) (x x)))
```

```
((λ (x) (x x)) (λ (x) (x x)))
```

β reduction may continue indefinitely (i.e., in non-terminating programs)

$$\Big\downarrow \beta$$

```
((λ (x) (x x)) (λ (x) (x x)))
```

$$\Big\downarrow \beta$$

```
((λ (x) (x x)) (λ (x) (x x)))
```

$$\Big\downarrow \beta$$

```
((λ (x) (x x)) (λ (x) (x x)))
```

$$\Big| \beta$$

```
((λ (x) (x x)) (λ (x) (x x)))
```

$\downarrow \beta$

```
((λ (x) (x x)) (λ (x) (x x)))
```

This specific program is known as Ω (Omega)

$\downarrow \beta$

```
((λ (x) (x x)) (λ (x) (x x)))
```

$\downarrow \beta$

```
((λ (x) (x x)) (λ (x) (x x)))
```

$\downarrow \beta$

((λ (x) (x x)) (λ (x) (x x)))

β

Ω is the smallest non-terminating program!

((λ (x) (x x)) (λ (x) (x x)))

Note how it reduces to itself in a single step!

β

((λ (x) (x x)) (λ (x) (x x)))

β

((λ (x) (x x)) (λ (x) (x x)))

β

Example

$$\big((\lambda(x)\ (x\ x))\ (\lambda(x)\ (x\ x))\big)$$

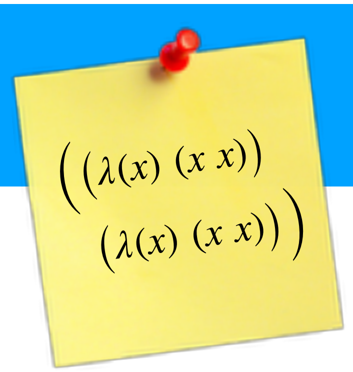
Evaluation with β reduction is nondeterministic!

```
(((λ (w) w) (λ (x) x)) ((λ (y) y) (λ (z) z)))
```

$$\downarrow \beta$$

```
((λ (x) x) ((λ (y) y) (λ (z) z)))
```

Example



Evaluation with β reduction is nondeterministic!

```
(((λ (w) w) (λ (x) x)) ((λ (y) y) (λ (z) z)))
```

β

```
((λ (x) x) ((λ (y) y) (λ (z) z)))
```

or!

β

```
(((λ (w) w) (λ (x) x)) (λ (z) z))
```

Exercise

Perform each possible β-reduction

```
((λ (x) ((λ (y) (x y)) x)) (λ (z) (z z)))
```

How many different β-reductions are possible from the above?

Exercise

```
((λ (x) ((λ (y) (x y)) x)) (λ (z) (z z)))
```

$\Big\downarrow \beta$

```
((λ (x) (x x)) (λ (z) (z z)))
```

Can reduce inner redex…

Exercise

```
((λ (x) ((λ (y) (x y)) x)) (λ (z) (z z)))
```

$$\Big\downarrow \beta$$

```
((λ (y) ((λ (z) (z z)) y)) (λ (z) (z z)))
```

Or the outer redex.

Exercise

```
((λ (x) ((λ (y) (x y)) x)) (λ (z) (z z)))
```

$$\Big\downarrow \beta$$

```
((λ (y) ((λ (z) (z z)) y)) (λ (z) (z z)))
```

Can’t reduce this since we don’t (yet) know about the particular value (function) z in call position.

Free Variables

We define the free variables of a lambda expression via the function FV:

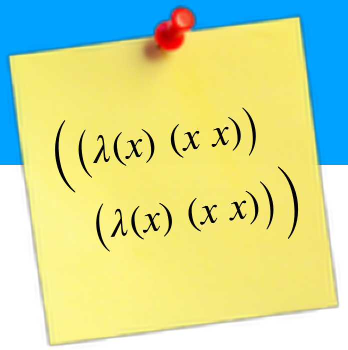
$$\textbf{FV} : \textbf{Exp} \to \mathscr{P}(\textbf{Var})$$

$$\textbf{FV}(x) \triangleq \{x\}$$

$$\textbf{FV}((\lambda\ (x)\ e_b)) \triangleq \textbf{FV}(e_b) \setminus \{x\}$$

$$\textbf{FV}(e_f\ e_a)) \triangleq \textbf{FV}(e_f)\ \cup\ \textbf{FV}(e_a)$$

Example



FV((x y)) = {x, y}

FV(((λ (x) x) y)) = {y}

FV(((λ (x) x) x)) = {x}

FV(((λ (y) ((λ (x) (z x)) x)))) = {z, x}

Exercise

What are the free variables of each of the following terms?

```
((λ (x) x) y)
```

```
((λ (x) (x x)) (λ (x) (x x)))
```

```
((λ (x) (z y)) x)
```

Exercise

What are the free variables of each of the following terms?

```
((λ (x) x) y)
```

{y}

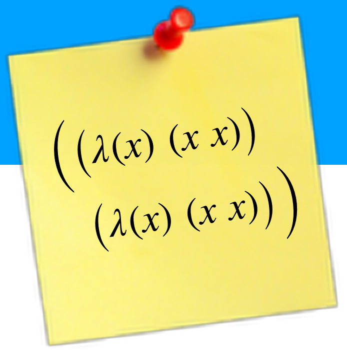
```
((λ (x) (x x)) (λ (x) (x x)))
```

{}

```
((λ (x) (z y)) x)
```

{x, y, z}

Example

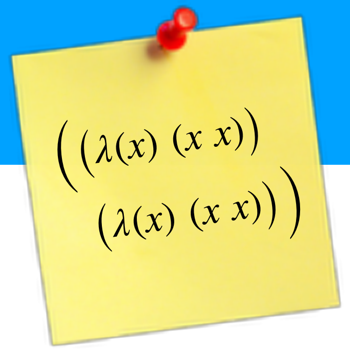


The problem with (naive) textual substitution

```
((λ (a) (λ (a) a)) (λ (b) b))
```

$$\Big\downarrow \beta$$

Example



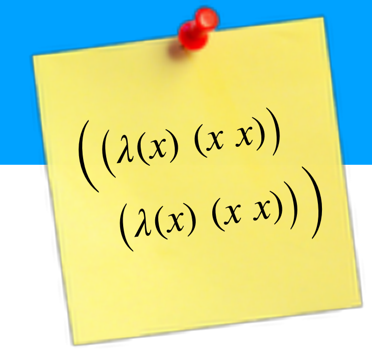
The problem with (naive) textual substitution

```
((λ (a) (λ (a) a)) (λ (b) b))
```

$$\Big\downarrow \beta$$

```
(λ (a) a)[a ← (λ (b) b)]
```

Example



The problem with (naive) textual substitution

```
((λ (a) (λ (a) a)) (λ (b) b))
```

$$\Big\downarrow \beta$$

```
(λ (a) (λ (b) b))
```

Capture-avoiding substitution

$$E_0[x \leftarrow E_1]$$

$$x[x \leftarrow E] = E$$

$$y[x \leftarrow E] = y \quad \text{where} \quad y \neq x$$

$$(E_0 \;\; E_1)[x \leftarrow E] = (E_0[x \leftarrow E] \;\; E_1[x \leftarrow E])$$

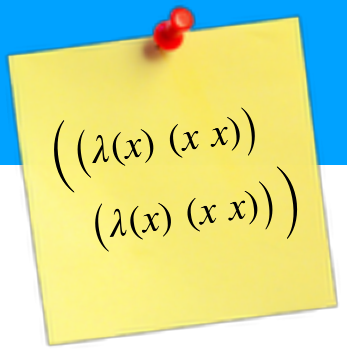
$$(\lambda \;\; (x) \;\; E_0)[x \leftarrow E] = (\lambda \;\; (x) \;\; E_0)$$

$$(\lambda \;\; (y) \;\; E_0)[x \leftarrow E] = (\lambda \;\; (y) \;\; E_0[x \leftarrow E])$$

where $y \neq x$ and $y \notin FV(E)$

β-reduction cannot occur when $y \in FV(E)$

Example



Capture-avoiding substitution

```
((λ (a) (λ (a) a)) (λ (b) b))
```

$$\Big\downarrow \beta$$

```
(λ (a) a)
```

Exercise

How can you beta-reduce the following expression using capture-avoiding substitution?

```
((λ (y)
    ((λ (z) (λ (y) (z y))) y))
 (λ (x) x))
```

Exercise

How can you beta-reduce the following expression using capture-avoiding substitution?

```
((λ (y)
    ((λ (z) (λ (y) (z y))) y))
 (λ (x) x))
```

$$\Big\downarrow \beta$$

```
((λ (z) (λ (y) (z y))) (λ (x) x))
```

Exercise

How can you beta-reduce the following expression using capture-avoiding substitution?

```
(λ (y) ((λ (z) (λ (y) z)) (λ (x) y)))
```

Exercise

How can you beta-reduce the following expression using capture-avoiding substitution?

`(λ (y) ((λ (z) (λ (y) z)) (λ (x) y)))`

You cannot! This redex would require:

`(λ (y) z)[z ← (λ (x) y)]`

(y is free here, so it would be captured)

Exercise

How can you beta-reduce the following expression using capture-avoiding substitution?

`(λ (y) ((λ (z) (λ (y) z)) (λ (x) y)))`

$\to_\alpha$ `(λ (y) ((λ (z) (λ (w) z)) (λ (x) y)))`

$\to_\beta$ `(λ (y) (λ (w) (λ (x) y)))`

Instead we alpha-convert first.

α - renaming

```
(λ (x) (λ (y) x))          (λ (a) (λ (b) a))
```

These two expressions are equivalent—they only differ by their variable names (`x = a`; `y = b`)

α-renaming

$$(\lambda\ (x)\ E_0) \quad \rightarrow_\alpha \quad (\lambda\ (y)\ E_0[x \leftarrow y])$$

$=_\alpha$

α renaming/conversions can be run backward, so you might think of it as an equivalence relation

α - renaming

α renaming/conversions can be used to implement capture-avoiding substitution

Rename variables that would break naive substitution!

```
((λ (x) (λ (x) x)) (λ (y) y))
```

α - renaming

α renaming/conversions can be used to implement capture-avoiding substitution

Rename variables that would break naive substitution!

```
((λ (x) (λ (x) x)) (λ (y) y))
```

Can’t perform naive substitution w/o capturing x.

α - renaming

α renaming/conversions can be used to implement capture-avoiding substitution

Rename variables that would break naive substitution!

```
((λ (x) (λ (x) x)) (λ (y) y))
```

Fix by α renaming to z

α - renaming

α renaming/conversions can be used to implement capture-avoiding substitution

Rename variables that would break naive substitution!

```
((λ (x) (λ (z) z)) (λ (y) y))
```

Fix by α renaming to z

α - renaming

α renaming/conversions can be used to implement capture-avoiding substitution

Rename variables that would break naive substitution!

```
((λ (x) (λ (z) z)) (λ (y) y))
```

Could now perform beta-reduction with naive substitution

η - reduction

$(\lambda\ (x)\ (E_0\ x)) \quad \rightarrow_\eta \quad E_0$ where $x \notin FV(E_0)$

η - expansion

$\texttt{E}_0 \quad \to_\eta \quad \texttt{(}\lambda\ \texttt{(x) (E}_0\ \texttt{x))}$ where $\texttt{x} \notin FV(\texttt{E}_0)$

Reduction

$$(\rightarrow) \;=\; (\rightarrow_\beta) \cup (\rightarrow_\alpha) \cup (\rightarrow_\eta)$$

$$(\rightarrow^*)$$

reflexive/transitive closure

Evaluation

$$\begin{tikzcd} \mathtt{E_0} \arrow[d, "*"'] \\ \mathtt{E_8} \arrow[d, "*"'] \\ ? \end{tikzcd}$$

Evaluation to *normal form*

$$E_0 \xrightarrow{*} (\lambda\ (x)\ \ldots)$$

Evaluation to *normal form*

$$E_0 \;\big\downarrow{}^{*}$$

`(λ (x) … (λ (z) ((a …) …)))`

In ***normal form***, no function position can be a lambda; this is to say: *there are no unreduced redexes left*!

Evaluation Strategy

$$
\begin{tikzcd}
& \mathtt{E}_0 \arrow[dl, "*"'] \arrow[dr, "*"] & \\
\mathtt{E}_1 & & \mathtt{E}_2
\end{tikzcd}
$$

Evaluation Strategy

```
(((λ (x) ((λ (y) y) x)) (λ (z) z))
```

$\rightarrow_\eta$ `((λ (y) y) (λ (z) z))`

$\rightarrow_\beta$ `(λ (z) z)`

Evaluation Strategy

```
((λ (x) ((λ (y) y) x)) (λ (z) z))
```

$\rightarrow_\beta$ `((λ (y) y) (λ (z) z))`

$\rightarrow_\beta$ `(λ (z) z)`

Evaluation Strategy

```
((λ (x) ((λ (y) y) x)) (λ (z) z))

        →β ((λ (x) x) (λ (z) z))

        →β (λ (z) z)
```

Confluence

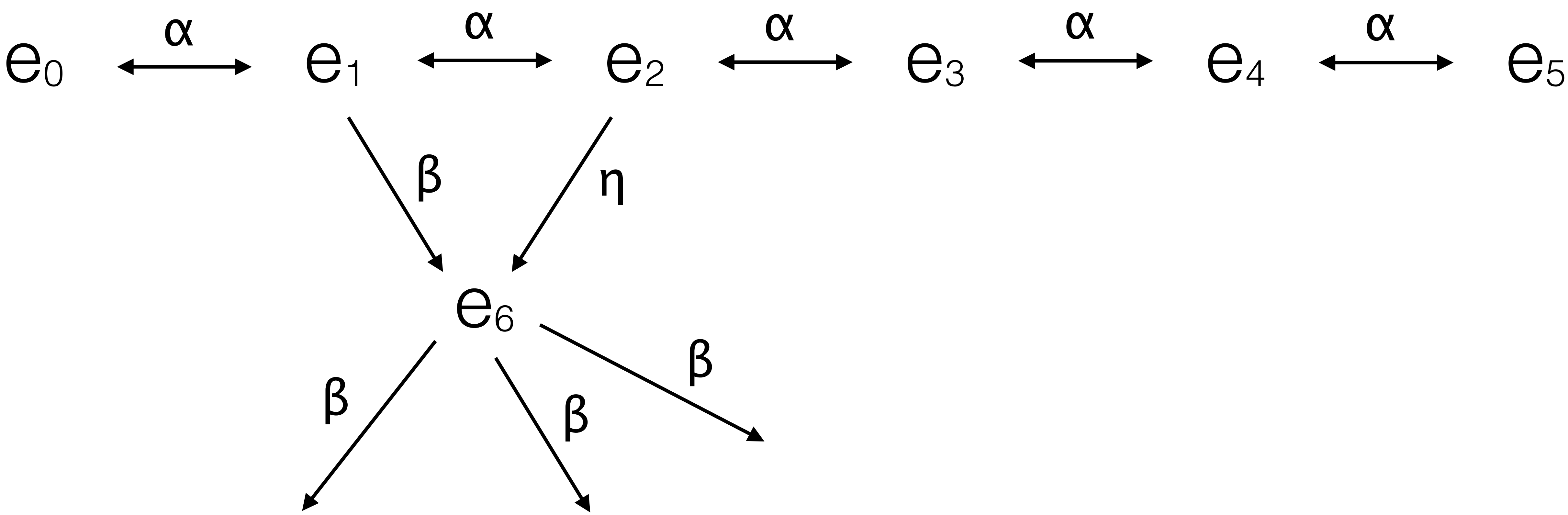
Diverging paths of evaluation must eventually join back together.

$$
\begin{tikzcd}
& E_0 \arrow[dl, "*"'] \arrow[dr, "*"] & \\
E_1 \arrow[dr, "*"'] & & E_2 \arrow[dl, "*"] \\
& E_3 &
\end{tikzcd}
$$

Church-Rosser Theorem

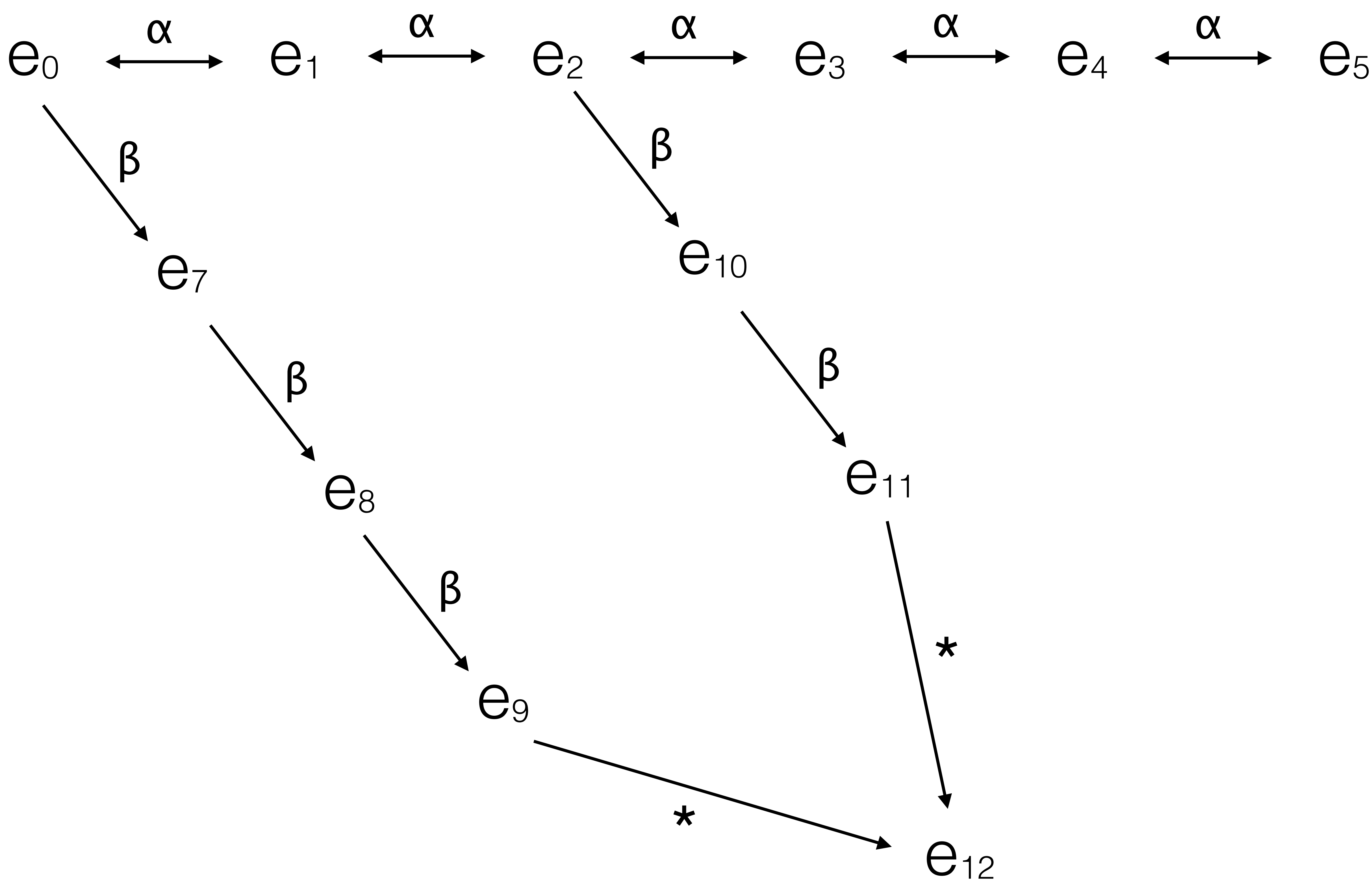
$$e_0 \xleftrightarrow{\alpha} e_1 \xleftrightarrow{\alpha} e_2 \xleftrightarrow{\alpha} e_3 \xleftrightarrow{\alpha} e_4 \xleftrightarrow{\alpha} e_5$$

$$
\begin{tikzcd}
e_0 \arrow[rr, leftrightarrow, "\alpha"] && e_1 \arrow[rr, leftrightarrow, "\alpha"] \arrow[dr, "\beta"] && e_2 \arrow[rr, leftrightarrow, "\alpha"] \arrow[dl, "\eta"] && e_3 \arrow[rr, leftrightarrow, "\alpha"] && e_4 \arrow[rr, leftrightarrow, "\alpha"] && e_5 \\
&&& e_6
\end{tikzcd}
$$


e0
α
e1
α
e2
α
e3
α
e4
α
e5
β
η
e6
β
β
β

$$
\begin{tikzcd}
e_0 \arrow[rr, leftrightarrow, "\alpha"] \arrow[dr, "\beta"] & & e_1 \arrow[rr, leftrightarrow, "\alpha"] & & e_2 \arrow[rr, leftrightarrow, "\alpha"] \arrow[dr, "\beta"] & & e_3 \arrow[rr, leftrightarrow, "\alpha"] & & e_4 \arrow[rr, leftrightarrow, "\alpha"] & & e_5 \\
& e_7 \arrow[dr, "\beta"] & & & & e_{10} \arrow[dr, "\beta"] & & & & & \\
& & e_8 \arrow[dr, "\beta"] & & & & e_{11} & & & & \\
& & & e_9 & & & & & & &
\end{tikzcd}
$$

Confluence (i.e., Church-Rosser Theorem)



Applicative evaluation order

Always evaluates the *innermost* leftmost redex first.

Normal evaluation order

Always evaluates the *outermost* leftmost redex first.

Applicative evaluation order

```
((λ (x) ((λ (y) y) x)) (λ (z) z))
```

Normal evaluation order

```
(((λ (x) ((λ (y) y) x)) (λ (z) z)) (λ (w) w))
```

Call-by-value (CBV) semantics

Applicative evaluation order, *but not under lambdas*.

Call-by-name (CBN) semantics

Normal evaluation order, *but not under lambdas*.

Exercise

Write a lambda term other than Ω which also does not terminate

(Hint: think about using some form of self-application)

Exercise

Write a lambda term other than Ω which also does not terminate

```
((λ (y) ((λ (x) (y x)) y))
 (λ (y) ((λ (x) (y x)) y)))
```

```
((λ (u) ((u u) u))
 (λ (u) ((u u) u)))
```

```
((λ (x) x)
 ((λ (u) (u u))
  (λ (u) (u u))))
```